1. Let $V$ be the subspace of $L^2(\mathbb{R})$ of all functions such that $f$ and $f'$ are piecewise continuous and vanish at $\pm\infty$. Define the two operators $D$ and $M$ by:

$$(Df)(t) = f'(t), \quad Mf(t) = tf(t), \quad f \in V.$$ 

(a) Simplify the expression $DM - MD$.

(b) Simplify the inner product $\langle (M - aI)f | (D - i\alpha I)f \rangle$ using the identity $\langle a + b | c + d \rangle = \langle a | c \rangle + \langle a | d \rangle + \langle b | c \rangle + \langle b | d \rangle$ and other basic properties of the inner product.

(c) Simplify the inner product $\langle (D - i\alpha I)f | (M - aI)f \rangle$, using the above idea.

(d) Verify the identity that was used in proving the Heisenberg Uncertainty Principle that is found at page 92 of the notes:

$$-\langle (M - aI)f | (D - i\alpha I)f \rangle - \langle (D - i\alpha I)f | (M - aI)f \rangle = \|f\|^2.$$

2. Let $f[n] = r^n$ for $n = 0, 1, \ldots, N - 1$ and $r \in \mathbb{C}$. Suppose that $r$ is not an $N$-th root of unity. Find its discrete Fourier transform. Supply all details.

3. Write the Discrete Fourier Transform for $N = 6$ in terms of the DFT with $N = 3$ as in the notes on page 99; that is, separate out the even and odd components. Next, write out the matrix factorization on page 100, equation (4.10) explicitly as well.