1. (a) Describe a subspace of $\mathbb{R}^{2\times2}$ that contains the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not the matrix $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
(b) If a subspace of $\mathbb{R}^{2\times2}$ contains $A$ and $B$, must it contain the identity matrix $I$?
(c) Describe a subspace of $\mathbb{R}^{2\times2}$ that contains no nonzero diagonal matrices.

**Answer:** (a) The smallest such subspace would be all scalar multiples of the matrix $A$, that is, the subspace \( \left\{ \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} : c \in \mathbb{R} \right\} \).
(b) Note: $I = A - B$.
(c) Let the subspace consist of all multiplies of the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

2. (a) Show that the set of all invertible matrices in $\mathbb{R}^{2\times2}$ is not a subspace.  
(b) Show that the set of all singular (=non-invertible) matrices in $\mathbb{R}^{2\times2}$ is not a subspace.

**Answer:** (a) The identity matrix $I$ is invertible, but $I - I = 0$ is not invertible.
(b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so neither matrix is invertible, but $I = A + B$.

3. If $A$ is any $5 \times 5$ invertible matrix, then its column space is ............... Why?

**Answer:** If $A$ is invertible, then it must have 5 pivots. In particular, the linear system $Ax = b$ is always solvable. Hence, its column space must be all of $\mathbb{R}^5$.

4. True or false (with a counterexample if false, and a reason if true):
   (a) The vectors $b$ that are not in the column space of a matrix $A$ form a subspace.
   (b) If the column space of $A$ contains only the zero vector, then $A$ is the zero matrix.
(c) The column space of $2A$ equals the column space of $A$.  

(d) The column space of $A - I$ equals the column space of $A$.

**Answer:** (a) FALSE: If $A$ is invertible and is in $\mathbb{R}^{2 \times 2}$, then its column space is $\mathbb{R}^2$. The set of vectors not in the column space then must be the empty set. This is NOT a subspace.

(b) TRUE: If the column space of $A$ contains only the zero vector, then every pivot of $A$ must be zero. Hence, no entry of $A$ can be non-zero. In other words, $A = 0$.

(c) TRUE: The column space of $2A$ equals the column space of $A$, since every column of $2A$ is a non-zero multiple of the corresponding column of $A$.

(d) FALSE: Take $A = I$. Then $A - I$ is the zero matrix.

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5. Construct a $3 \times 3$ matrix whose column space contains the vectors $(1, 1, 0)$ and $(1, 0, 1)$ but not the vector $(1, 1, 1)$.

**Answer:** The easiest example is to take $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. It is easy to verify that the vector $(1, 1, 1)$ is not a linear combination of $(1, 1, 0)$ and $(1, 0, 1)$. 

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