Show all your work on the exam paper, legibly and in detail, to receive full credit. The use of a calculator or any other electronic device is prohibited. You may only use techniques discussed to date in class. You must simplify all answers unless you are explicitly instructed not to.

1. For both parts a) and b), write out the form of the partial fraction decomposition. Do NOT find the numerical values of the coefficients.

a. \( \frac{x^3 + 2}{x^4 + x^2} = \frac{x^3 + 2}{x^2(x^2 + 1)} \)

\[ = \frac{A}{x} + \frac{B}{x^2} + \frac{C x + D}{x^2 + 1} \]

b. \( \frac{x^3 - 3x^2 + 2}{x^2 - 3x - 4} \)

Hint: Don’t forget to examine the highest powers of \( x \) in the numerator and denominator.

\[ \frac{x^3 - 3x^2 - 4x + 2}{x^2 - 3x - 4} \]

\[ = \frac{x^3 - 3x^2 - 4x + 2}{4x + 2} \]

\[ = x + \frac{4x + 2}{x^2 - 3x - 4} = x + \frac{4x + 2}{(x-4)(x+1)} \]

\[ = x + \frac{A}{x-4} + \frac{B}{x+1} \]
2. (a) (10 pts) Find the partial fraction expansion for the following function. To get full credit, you must find the numerical values of the coefficients in the partial fraction expansion.

\[
\frac{3x^2 + 5x - 4}{(2x - 1)(x - 1)(x + 1)} = \frac{A}{2x - 1} + \frac{B}{x - 1} + \frac{C}{x + 1}
\]

\[
3x^2 + 5x - 4 = A(x - 1)(x + 1) + B(2x - 1)(x + 1) + C(2x - 1)(x - 1)
\]

\[
\begin{align*}
    x = 1: & \quad 3 + 5 - 4 = 0 + B(1)(2) + 0 \quad \Rightarrow \quad 4 = 2B \quad \Rightarrow \quad B = 2 \\
    x = -1: & \quad 3 - 5 - 4 = 0 + 0 + C(-3)(-2) \quad \Rightarrow \quad -6 = 6C \quad \Rightarrow \quad C = -1 \\
    x = \frac{1}{2}: & \quad \frac{3}{4} + \frac{5}{2} - 4 = A\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right) + 0 + 0 \\
\end{align*}
\]

\[
\frac{3 + 10 - 16}{4} = -\frac{3}{4}A \quad \Rightarrow \quad -3 = -3A \quad \Rightarrow \quad A = 1
\]

(b) (5 pts) Evaluate the following integral:

\[
\int \frac{3x^2 + 5x - 4}{(2x - 1)(x - 1)(x + 1)} \, dx
\]

\[
= \int \frac{1}{2x - 1} \, dx + \int \frac{2}{x - 1} \, dx + \int \frac{-1}{x + 1} \, dx
\]

\[
= \frac{1}{2} \ln \left|2x - 1\right| + 2 \ln \left|x - 1\right| - \ln \left|x + 1\right| + C
\]
3. (10 pts) Consider the differential equation:

\[
\frac{dy}{dx} + 2y = 3e^{2x}
\]

You are told that there is a solution to this equation of the form

\[
y(x) = Ae^{2x}
\]

where \( A \) is a constant. What does \( A \) have to be for \( y(x) \) to be a solution.

\[
y = Ae^{2x} \implies \frac{dy}{dx} = 2Ae^{2x}
\]

If \( \frac{dy}{dx} + 2y = 3e^{2x} \), then

\[
2Ae^{2x} + 2Ae^{2x} = 3e^{2x}
\]

\[
4Ae^{2x} = 3e^{2x}
\]

\[
4A = 3
\]

\[
A = \frac{3}{4}
\]
4. (a) (10 pts) Find the general solution to the following differential equation by separation of variables.

(b) (5 pts) Express the family of solutions as explicit functions of $x$.

\[ e^{-y} \sin x - \frac{dy}{dx} \cos^2 x = 0 \]

\[ e^{-y} \sin x = \frac{dy}{dx} \cos^2 x \]

\[ \frac{\sin x}{\cos^2 x} \frac{dx}{dy} = e^y \]

\[ \int \frac{1}{\cos x} \frac{\sin x}{\cos x} \frac{dx}{dy} = \int e^y \frac{dy}{dy} \]

\[ \int \sec x \tan x \, dx = \int e^y \, dy \]

\[ \sec x + C = e^y \]

\[ y = \ln (\sec x + C) \]
5. (a) (10 pts) Solve the following differential equation:

\[ \frac{dy}{dx} + 2xy = x \]

Using separation of variables

\[ \frac{dy}{dx} = x - 2xy = x(1-2y) \]

\[ \text{If } y \neq \frac{1}{2}, \quad \frac{dy}{1-2y} = x \frac{dx}{x} \implies \int \frac{dy}{1-2y} = \int x \, dx \]

\[ -\frac{1}{2} \ln |1-2y| = \frac{1}{2} x^2 + C \quad \Rightarrow \quad \ln |1-2y| = -x^2 + C \]

\[ 1-2y = \pm e^{-x^2 + C} = \pm e^C e^{-x^2} = Ce^{-x^2} \]

\[ 2y = 1-Ce^{-x^2} \quad \Rightarrow \quad y = \frac{1}{2} + Ce^{-x^2} \]

If C=0, \( y = \frac{1}{2} \) which is also a solution

(b) (5 pts) Find the particular solution to the differential equation given above, satisfying the initial condition:

\[ y \left( \sqrt{\ln(2)} \right) = 4 \]

\[ 4 = \frac{1}{2} + Ce^{-\left(\sqrt{\ln(2)}\right)^2} \]

\[ 4 = \frac{1}{2} + Ce^{-\ln 2} = \frac{1}{2} + Ce^{\ln \frac{1}{2}} \]

\[ 4 = \frac{1}{2} + \frac{1}{2} C \]

\[ \frac{7}{2} = \frac{1}{2} C \quad \Rightarrow \quad C = 7 \]

\[ y = \frac{1}{2} + 7e^{-x^2} \]
5. (a) (10 pts) Solve the following differential equation:
\[
\frac{dy}{dx} + 2xy = x \\
\text{Using integrating factors}
\]
\[
\mu(x) = e^{\int 2x \, dx} = e^{x^2}
\]
\[
\frac{dy}{dx} e^{x^2} + 2xy e^{x^2} = x e^{x^2}
\]
\[
\frac{d}{dx} (e^{x^2} y) = x e^{x^2} \implies \int \frac{d}{dx} (e^{x^2} y) \, dx = \int x e^{x^2} \, dx
\]
\[
e^{x^2} y = \frac{1}{2} x e^{x^2} + C
\]
\[
\boxed{y = \frac{1}{2} + C e^{-x^2}}
\]

(b) (5 pts) Find the particular solution to the differential equation given above, satisfying the initial condition:
\[
y \left( \sqrt{\ln(2)} \right) = 4
\]
6. (10 pts) Recall that the exponential growth model is of the form:

\[ \frac{dy}{dt} = ky \quad (k > 0), \quad y(0) = y_0 \]

where \( y = y(t) \) represents the population at time \( t \) and \( y_0 \) is the initial population, and that its solution is \( y(t) = y_0 e^{kt} \).

A scientist observes that when 10 fruit flies are placed in a breeding container the population doubles in size in 3 days. Assuming that the experiment can be modeled with the exponential growth model, find the growth constant \( k \) and determine how long it takes before the population of fruit flies triples in size. Note: You may leave \( \ln \) in your answer.

Given \( y_0 = 10, \quad y(3) = 20 \)

\[ 20 = 10 e^{k(3)} \quad \Rightarrow \quad 2 = e^{3k} \]

\[ \ln 2 = \ln e^{3k} = 3k \quad \Rightarrow \quad k = \frac{\ln 2}{3} \]

So \( y(t) = 10 e^{\frac{\ln 2}{3} t} = 10 (e^{\ln 2})^{\frac{t}{3}} = 10 \cdot 2^{\frac{t}{3}} \)

Must find \( t \) that makes \( y = 30 \)

\[ 30 = 10 \cdot 2^{\frac{t}{3}} \]

\[ 3 = 2^{\frac{t}{3}} \quad \Rightarrow \quad \ln 3 = \ln 2^{\frac{t}{3}} = \frac{t}{3} \ln 2 \]

\[ t = \frac{3 \ln 3}{\ln 2} \]
7. At time $t = 0$, a tank contains 25 ounces of salt dissolved in 50 gallons of water. Then salt water containing 4 ounces of salt per gallon enters the tank at a rate of 2 gal/min, and the mixed solution is drained from the tank at the rate of 2 gal/min.

(a)(10 pts) What is the differential equation which describes the rate of change of $S(t)$, the amount of salt at any time $t$?

$$\frac{dS}{dt} = \text{rate in} - \text{rate out} = 8 - \frac{1}{25} S$$

rate in: $\frac{4 \text{ oz}}{\text{gal}} \cdot \frac{2 \text{ gal}}{\text{min}} = \frac{8 \text{ oz}}{\text{min}}$

rate out: $\frac{S}{50} \frac{\text{oz}}{\text{gal}} \cdot \frac{2 \text{ gal}}{\text{min}} = \frac{1}{25} S$

(b)(5 pts) What is the general solution to the differential equation from (a)?

$$\frac{dS}{dt} = 8 - \frac{1}{25} S \Rightarrow 25 \frac{dS}{dt} = 200 - S \Rightarrow \int \frac{dS}{200 - S} = \int \frac{1}{25} dt$$

$$- \ln |200 - S| = \frac{1}{25} t + C \Rightarrow |200 - S| = e^{\frac{t}{25} + C} = e^C e^{\frac{-t}{25}}$$

$$200 - S = C e^{\frac{-t}{25}}$$

$$S = 200 - C e^{\frac{-t}{25}} \Rightarrow S(t) = 200 + C e^{\frac{-t}{25}}$$

(c)(5 pts) Which particular solution satisfies the initial condition stated in the problem?

Given $S(0) = 25$

$$25 = 200 + Ce^0 = 200 + C$$

$$C = -175$$

$$S(t) = 200 - 175 e^{\frac{-t}{25}}$$
(76) With integrating factors:

\[
\frac{ds}{dt} = 8 - \frac{1}{25} s
\]

\[
\frac{ds}{dt} + \frac{1}{25} s = 8
\]

\[
\mu(t) = e^{\int \frac{1}{25} dt} = e^{\frac{1}{25} t}
\]

\[
\frac{ds}{dt} e^{\frac{t}{25}} + \frac{1}{25} s e^{\frac{t}{25}} = 8 e^{\frac{t}{25}}
\]

\[
\frac{d}{dt} (e^{\frac{t}{25}} s) = 8 e^{\frac{t}{25}}
\]

\[
\int \frac{d}{dt} (e^{\frac{t}{25}} s) dt = \int 8 e^{\frac{t}{25}} dt
\]

\[
e^{\frac{t}{25}} s = 8 e^{\frac{t}{25}} (25) + C
\]

\[
S(t) = 200 + Ce^{-\frac{t}{25}}
\]