Show all your work on the exam paper, legibly and in detail, to receive full credit. The use of a calculator or any other electronic device is prohibited. You may only use techniques discussed to date in class. You must simplify all answers unless you are explicitly instructed not to.

Some useful formulas:

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{k=1}^{n} k^3 = \left[ \frac{n(n+1)}{2} \right]^2
\]

1. (8 pts) State a corresponding integration formula.

\[
\frac{d}{dx} \left[ xe^x \right] = (x+1)e^x
\]

\[
\int (x+1)e^x \, dx = xe^x + C
\]

2. (12 pts) Given that \( f(x) = \begin{cases} x^2, & x \leq 3 \\ 9, & x > 3 \end{cases} \) evaluate \( \int_{0}^{5} f(x) \, dx \).

\[
\int_{0}^{5} f(x) \, dx = \int_{0}^{3} x^2 \, dx + \int_{3}^{5} 9 \, dx
\]

\[
= \frac{1}{3} x^3 \bigg|_{0}^{3} + 9x \bigg|_{3}^{5} = \frac{1}{3} (3^3 - 0^3) + 9(5-3)
\]

\[
= \frac{1}{3} (27) + 9(2) = 9 + 18 = 27
\]
3. Evaluate the following indefinite integrals (anti-derivatives).

a. (6 pts) \( \int \frac{x^2 + 3x + 1}{x^2} \, dx \)

\[
\int \left( 1 + \frac{3}{x} + x^{-2} \right) \, dx = x + 3 \ln |x| - \frac{1}{x} + C
\]

b. (6 pts) \( \int \sin (\tan \theta) \sec^2 \theta \, d\theta \)

\[
u = \tan \theta \quad \Rightarrow \quad du = \sec^2 \theta \, d\theta
\]

\[
\int \sin u \, du = -\cos u + C
\]

\[
= -\cos (\tan \theta) + C
\]

c. (4 pts) \( \int \frac{e^t}{\sqrt{1-e^{2t}}} \, dt \)

\[
u = e^t \quad \Rightarrow \quad du = e^t \, dt
\]

\[
\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C
\]

\[
= \arcsin (e^t) + C
\]
4. Evaluate the following definite integrals.

a. (10 pts) \[ \int_{e^5}^{e} \frac{dx}{x \ln x} \]

\[ u = \ln x \Rightarrow du = \frac{1}{x} \, dx \]

\[ x = e^5 \Rightarrow u = \ln e^5 = 5 \]

\[ x = e \Rightarrow u = \ln e = 1 \]

\[ \int_{1}^{5} \frac{du}{u} = \ln |u| \bigg|_{1}^{5} = \ln 5 - \ln 1 = \ln 5 \]

b. (6 pts) \[ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cos \theta + \csc^2 \theta) \, d\theta \]

\[ = \sin \theta \bigg[ \frac{\pi}{2} - \cot \theta \bigg]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \]

\[ = (\sin \frac{\pi}{2} - \sin \frac{\pi}{4}) - (\cot \frac{\pi}{2} - \cot \frac{\pi}{4}) \]

\[ = (1 - \frac{\sqrt{2}}{2}) - (0 - 1) = 1 - \frac{\sqrt{2}}{2} + 1 = 2 - \frac{\sqrt{2}}{2} \]
5. Find the exact area under the curve \( f(x) = 2x + 3 \) and over the interval \([0,2]\) using the specified technique.

a. (14 points) Use the limit of a Riemann sum with \( x^*_k \) as the right endpoint of each subinterval. Make all of your subintervals of equal length. **To get credit for this part, you MUST use the Riemann sum method.**

\[
a = 0 \quad b = 2 \quad \Delta x = \frac{b-a}{n} = \frac{2}{n}
\]

\[
x^*_k = a + k \Delta x = \frac{2k}{n}
\]

\[
f(x^*_k) = 2 \left( \frac{2k}{n} \right) + 3 = \frac{4k}{n} + 3
\]

\[
\frac{2}{n} \sum_{k=1}^{n} \left( \frac{4k}{n} + 3 \right) \left( \frac{2}{n} \right) = \frac{2}{n} \left[ \frac{4}{n} \sum_{k=1}^{n} k + 3 \sum_{k=1}^{n} 1 \right]
\]

\[
= \frac{2}{n} \left[ \frac{4}{n} \frac{n(n+1)}{2} + 3n \right]
\]

\[
= 4 \left( \frac{n+1}{n} \right) + 6
\]

\[
\lim_{n \to \infty} \left[ 4 \left( \frac{n+1}{n} \right) + 6 \right] = 4(1) + 6 = 10
\]
5. Find the exact area under the curve \( f(x) = 2x + 3 \) and over the interval \([0,2]\) using the specified technique.

b. (8 points) Use appropriate formulas from geometry. You must show a graph.

\[
\text{Area of triangle: } \frac{1}{2} (2)(4) = 4
\]
\[
\text{Area of rectangle: } (2)(3) = 6
\]
\[
\text{Area under curve: } 4 + 6 = 10
\]

c. (10 points) Use the fundamental theorem of calculus.

\[
\int_{0}^{2} (2x + 3) \, dx
\]
\[
= \left[ x^2 + 3x \right]_{0}^{2} = (2^2 - 0^2) + 3(2 - 0)
\]
\[
= 4 + 6 = 10
\]
6. (6 points) Write the expression in sigma notation but do not evaluate.

\[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\]

\[\sum_{k=1}^{5} (-1)^{k+1} \frac{1}{k}\]

7. Let \(F(x) = \int_{0}^{x} \frac{t+5}{t^2+7} dt\)

a. (5 points) Find \(F'(x)\)

\[F'(x) = \frac{d}{dx} \int_{0}^{x} \frac{t+5}{t^2+7} dt = \frac{x+5}{x^2+7}\]

b. (5 points) \(F\) has one critical point. Find it, and state whether it is a maximum, minimum, or neither. Explain your answer (guesses are not enough).

\[F'(x) = 0 \Rightarrow x+5 = 0 \Rightarrow x = -5\]

**First Derivative Test**:

\[F'(-6) = \frac{-6+5}{(-6)^2+7} < 0 \Rightarrow F\ is\ decreasing\ on\ (-\infty, -5)\]

\[F'(0) = \frac{0+5}{0^2+7} > 0 \Rightarrow F\ is\ increasing\ on\ (-5, +\infty)\]

So \(F\ has\ a\ minimum\ at\ x = -5\)