Lecture 1: Brownian motion, martingales and Markov processes

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Outline

2. Stopping times. Martingales.
4. Itô’s formula and applications.
5. Stochastic differential equations.
6. Introduction to Malliavin calculus.
Multivariate normal distribution

- A random vector \( X = (X_1, \ldots, X_n) \) has the multivariate normal distribution \( N(\mu, \Sigma) \), if its characteristic function is

\[
E \left( e^{i \langle u, x \rangle} \right) = \exp \left( i \langle u, \mu \rangle - \frac{1}{2} u^T \Sigma u \right), \quad u \in \mathbb{R}^n,
\]

where \( \mu \in \mathbb{R}^n \) and \( \Sigma \) is an \( n \times n \) symmetric and nonnegative definite matrix.

- \( \mu = (E(X_1), \ldots, E(X_n)) \)

- \( \Sigma_{ij} = \text{Cov}(X_i, X_j) \)

- If \( X \) has the \( N(\mu, \Sigma) \) distribution, then \( Y = AX + b \), where \( A \) is an \( m \times n \) matrix and \( b \in \mathbb{R}^m \), has the \( N(A\mu + b, A\Sigma A^T) \) distribution.
If $\Sigma$ is nonsingular, then $X$ has a density given by

$$f(x) = (2\pi)^{-\frac{n}{2}} (\det \Sigma)^{-\frac{1}{2}} \exp \left( -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right).$$
A stochastic process $X = \{X_t, t \geq 0\}$ is a family of random variables $X_t : \Omega \to \mathbb{R}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. The finite-dimensional marginal distributions of the process $X$ are called the finite-dimensional marginal distributions of the process $X$.
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The probabilities on $\mathbb{R}^n$, $n \geq 1$,

$$P_{t_1, \ldots, t_n} = P \circ (X_{t_1}, \ldots, X_{t_n})^{-1}$$

where $0 \leq t_1 < \cdots < t_n$, are called the finite-dimensional marginal distributions of the process $X$. 
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For every $\omega \in \Omega$, the mapping

$$t \rightarrow X_t(\omega)$$

is called a trajectory of the process $X$. 
Theorem (Kolmogorov’s extension theorem)

Consider a family of probability measures

\[ \{P_{t_1, \ldots, t_n}, \ 0 \leq t_1 < \cdots < t_n, n \geq 1\} \]

such that:

(i) \( P_{t_1, \ldots, t_n} \) is a probability on \( \mathbb{R}^n \).

(ii) (Consistence condition): If \( \{t_{k_1} < \cdots < t_{k_m}\} \subset \{t_1 < \cdots < t_n\} \), then \( P_{t_{k_1}, \ldots, t_{k_m}} \) is the marginal of \( P_{t_1, \ldots, t_n} \), corresponding to the indexes \( k_1, \ldots, k_m \).

Then, there exists a stochastic process \( \{X_t, t \geq 0\} \) defined in some probability space \( (\Omega, \mathcal{F}, P) \), which has the family \( \{P_{t_1, \ldots, t_n}\} \) as finite-dimensional marginal distributions.

- Take \( \Omega \) as the set of all functions \( \omega : [0, \infty) \rightarrow \mathbb{R} \), \( \mathcal{F} \) the \( \sigma \)-algebra generated by cylindrical sets, extend the probability from cylindrical sets to \( \mathcal{F} \), and set \( X_t(\omega) = \omega(t) \).
Gaussian processes

- $X = \{X_t, t \geq 0\}$ is called Gaussian if all its finite-dimensional marginal distributions are multivariate normal.

- The law of a Gaussian process is determined by the mean function $E(X_t)$ and the covariance function

$$\text{Cov}(X_t, X_s) = E((X_t - E(X_t))(X_s - E(X_s))).$$

- Suppose $\mu : \mathbb{R}_+ \to \mathbb{R}$, and $\Gamma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is symmetric and nonnegative definite:

$$\sum_{i,j=1}^{n} \Gamma(t_i, t_j) a_i a_j \geq 0, \quad \forall t_i \geq 0, a_i \in \mathbb{R}.$$

Then there exists a Gaussian process with mean $\mu$ and covariance function $\Gamma$. 
Equivalent processes

- Two processes, $X$, $Y$ are equivalent (or $X$ is a version of $Y$) if for all $t \geq 0$,

\[ P\{X_t = Y_t\} = 1. \]
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  \[ P\{X_t = Y_t\} = 1. \]

- Two equivalent processes may have quite different trajectories. For example, the processes $X_t = 0$ for all $t \geq 0$ and
  \[ Y_t = \begin{cases} 
  0 & \text{if } \xi \neq t \\
  1 & \text{if } \xi = t 
  \end{cases} \]
  where $\xi \geq 0$ is a continuous random variable, are equivalent, because
  $P(\xi = t) = 0$, but their trajectories are different.
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- Two processes $X$ and $Y$ are said to be indistinguishable if

$$X_t(\omega) = Y_t(\omega)$$

for all $t \geq 0$ and for all $\omega \in \Omega^*$, with $P(\Omega^*) = 1$.

Exercise: Two equivalent processes with right-continuous trajectories are indistinguishable.
Theorem (Kolmogorov’s continuity theorem)

Suppose that $X = \{X_t, t \in [0, T]\}$ satisfies

$$E(|X_t - X_s|^\beta) \leq K|t - s|^{1+\alpha},$$

for all $s, t \in [0, T]$, and for some constants $\beta, \alpha > 0$. Then, there exists a version $\tilde{X}$ of $X$ such that, if $\gamma < \alpha/\beta$,

$$|\tilde{X}_t - \tilde{X}_s| \leq G_\gamma |t - s|^\gamma$$

for all $s, t \in [0, T]$, where $G_\gamma$ is a random variable.

- The trajectories of $\tilde{X}$ are Hölder continuous of order $\gamma$ for any $\gamma < \alpha/\beta$. 
Sketch of the proof:

(i) Suppose $T = 1$. Take $\gamma < \alpha / \beta$ and set $D_n = \{ \frac{k}{2^n}, 0 \leq k \leq 2^n \}$ and $\mathcal{D} = \bigcup_{n \geq 1} D_n$. From Chebychev’s inequality,

$$P(\max_{1 \leq k \leq 2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \geq 2^{-\gamma n}) \leq \sum_{k=1}^{2^n} P(|X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}| \geq 2^{-\gamma n})$$

$$\leq \sum_{k=1}^{2^n} 2^{\gamma \beta n} E[|X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}|^\beta]$$

$$\leq K 2^{-n(\alpha - \gamma \beta)}.$$

Because this series of probabilities is convergent, from the Borel-Cantelli lemma, there is a set $\Omega^* \in \mathcal{F}$ with $P(\Omega^*) = 1$ such that for all $\omega \in \Omega^*$, there exists $N(\omega)$ with

$$|X_{\frac{k}{2^n}}(\omega) - X_{\frac{k-1}{2^n}}(\omega)| < 2^{-\gamma n}, \quad \forall n \geq N(\omega), \quad \forall 1 \leq k \leq 2^n.$$
(ii) Suppose that \( s, t \in \mathcal{D} \) are such that

\[ |s - t| \leq 2^{-n}, \quad n \geq N. \]

Then, there exists two increasing sequences \( s_k \in \mathcal{D}_k \) and \( t_k \in \mathcal{D}_k \), \( k \geq n \), converging to \( s \) and \( t \) respectively, and such that

\[ |s_{k+1} - s_k| \leq 2^{-(k+1)}, \quad |t_{k+1} - t_k| \leq 2^{-(k+1)} \]

and \( |s_n - t_n| \leq 2^{-n} \). Then, from the decomposition

\[ X_s - X_t = \sum_{i=n}^{\infty} (X_{s_{i+1}} - X_{s_i}) + (X_{s_n} - X_{t_n}) + \sum_{i=n}^{\infty} (X_{t_i} - X_{t_{i+1}}) \]

we obtain

\[ |X_t - X_s| \leq \frac{2}{1 - 2^{-\gamma}} 2^{-\gamma n}. \]

This implies that the paths \( t \rightarrow X_t(\omega) \) are \( \gamma \)-Hölder on \( \mathcal{D} \) for all \( \omega \in \Omega^* \), which allows us to conclude the proof. \( \square \)
Brownian motion

A stochastic process $B = \{B_t, t \geq 0\}$ is called a Brownian motion if:

i) $B_0 = 0$ almost surely.

ii) *Independent increments*: For all $0 \leq t_1 < \cdots < t_n$ the increments $B_{t_n} - B_{t_{n-1}}, \ldots, B_{t_2} - B_{t_1}$, are independent random variables.

iii) If $0 \leq s < t$, the increment $B_t - B_s$ has the normal distribution $N(0, t - s)$.

iv) With probability one, $t \to B_t(\omega)$ is continuous.
Proposition

Properties i), ii), iii) are equivalent to:

(*) \( B \) is a Gaussian process with mean zero and covariance

\[ \Gamma(s, t) = \min(s, t). \]
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\]

Proof:

a) Suppose i), i) and iii). The distribution of \((B_{t_1}, \ldots, B_{t_n})\), for \(0 < t_1 < \cdots < t_n\), is normal, because this vector is a linear transformation of \((B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})\) which has independent and normal components.

The mean is zero, and for \(s < t\), the covariance is

\[
E(B_sB_t) = E(B_s(B_t - B_s + B_s)) = E(B_s(B_t - B_s)) + E(B_s^2) = s.
\]

b) The converse is also easy to show. \(\square\)
1. The function $\Gamma(s, t) = \min(s, t)$ is symmetric and nonnegative definite because it can be written as

$$\min(s, t) = \int_0^\infty 1_{[0,s]}(r)1_{[0,t]}(r)dr,$$

so

$$\sum_{i,j=1}^n a_i a_j \min(t_i, t_j) = \sum_{i,j=1}^n a_i a_j \int_0^\infty 1_{[0,t_i]}(r)1_{[0,t_j]}(r)dr$$

$$= \int_0^\infty \left[ \sum_{i=1}^n a_i 1_{[0,t_i]}(r) \right]^2 dr \geq 0.$$

Therefore, by Kolmogorov’s extension theorem there exists a Gaussian process $B$ with zero mean and covariance function $\min(s, t)$. 

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2. The process $B$ satisfies

$$E \left[ (B_t - B_s)^{2k} \right] = \frac{(2k)!}{2^k k!} (t - s)^k, \quad s \leq t$$

for any $k \geq 1$, because the distribution of $B_t - B_s$ is $N(0, t - s)$.

3. Therefore, by the Kolmogorov’s continuity theorem, there exist a version $\tilde{B}$ of $B$, such that $\tilde{B}$ has Hölder continuous trajectories of order $\gamma$ for any $\gamma < \frac{k-1}{2k}$ on any interval $[0, T]$. This implies that the paths are $\gamma$-Hölder on $[0, T]$ for any $\gamma < \frac{1}{2}$ and for any $T > 0$. 
Second construction of Brownian motion

Fix $T > 0$.

(i) $\{e_n, n \geq 0\}$ is an orthonormal basis of $L^2([0, T])$.

(ii) $\{Z_n, n \geq 0\}$ are independent $N(0, 1)$ random variables.

Then, as $N \to \infty$,

$$
\sup_{0 \leq t \leq T} \left| \sum_{n=0}^{N} Z_n \int_{0}^{t} e_n(s) ds - B_t \right| \xrightarrow{a.s., L^2} 0.
$$

Notice that

$$
E \left[ \left( \sum_{n=0}^{N} Z_n \int_{0}^{t} e_n(r) dr \right) \left( \sum_{n=0}^{N} Z_n \int_{0}^{s} e_n(r) dr \right) \right]
= \sum_{n=0}^{N} \left( \int_{0}^{t} e_n(r) dr \right) \left( \int_{0}^{s} e_n(r) dr \right)
= \sum_{n=0}^{N} \left\langle \mathbf{1}_{[0,t]}, e_n \right\rangle_{L^2([0,T])} \left\langle \mathbf{1}_{[0,s]}, e_n \right\rangle_{L^2([0,T])} \xrightarrow{N \to \infty} \left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{L^2([0,T])} = s \wedge t.
$$
In particular, if $T = 2\pi$, $e_0(t) = \frac{1}{\sqrt{2\pi}}$ and $e_n(t) = \frac{1}{\sqrt{\pi}} \cos(nt/2)$, for $n \geq 1$, we obtain the Paley-Wiener representation of Brownian motion:

$$B_t = Z_0 \frac{t}{\sqrt{2\pi}} + \frac{2}{\sqrt{\pi}} \sum_{n=1}^{\infty} Z_n \frac{\sin(nt/2)}{n}, \quad t \in [0, 2\pi].$$

In order to use this formula to get a simulation of Brownian motion, we have to choose some number $M$ of trigonometric functions and a number $N$ of discretization points.
Let \( \{\xi_k, 1 \leq k \leq n\} \) be independent and identically distributed random variables with zero mean and variance one.

Define \( S_n(0) = 0, \)

\[
S_n\left(\frac{kT}{n}\right) = \sqrt{T} \frac{\xi_1 + \cdots + \xi_k}{\sqrt{n}}, \quad k = 1, \ldots, n
\]

and extend \( S_n(t) \) to \( t \in [0, T] \) by linear interpolation.

**Donsker Invariance Principle**: The law of the random walk \( S_n \) on \( C([0, T]) \) converges to the Wiener measure, which is the law of the Brownian motion. That is, that for any continuous and bounded function \( \varphi : C([0, T]) \to \mathbb{R} \),

\[
E(\varphi(S_n)) \overset{n \to \infty}{\to} E(\varphi(B)),
\]
Simulations of Brownian motion
Basic properties

1. *Selfsimilarity* :
   For any $a > 0$, the process $\{a^{-\frac{1}{2}} B_{at}, t \geq 0\}$ is also a Brownian motion.
2. For any $h > 0$, the process $\{B_{t+h} - B_h, t \geq 0\}$ is a Brownian motion.

3. The process $\{-B_t, t \geq 0\}$ is a Brownian motion.

4. Almost surely $\lim_{t \to \infty} \frac{B_t}{t} = 0$ and the process

$$X_t = \begin{cases} \frac{tB_{1/t}}{t}, & t > 0 \\ 0, & t = 0 \end{cases}$$

is a Brownian motion.

5. $P(\sup_{s,t \in [0,1]} \frac{|B_t - B_s|}{\sqrt{|t-s|}} = +\infty) = 1$.

6. $P(\sup_{t \geq 0} B_t = +\infty, \inf_{t \geq 0} B_t = -\infty) = 1$.

7. Almost surely the paths of $B$ are not differentiable at any point $t \geq 0$. 
Fix a time interval $[0, t]$ and consider a partition

$$\pi = \{0 = t_0 < t_1 < \cdots < t_n = t\}.$$ 

Define $\Delta t_k = t_k - t_{k-1}$, $\Delta B_k = B_{t_k} - B_{t_{k-1}}$ and $|\pi| = \max_{1 \leq k \leq n} \Delta t_k$.

**Proposition**

The following convergence holds in $L^2$:

$$\lim_{|\pi| \to 0} \sum_{k=1}^{n} (\Delta B_k)^2 = t.$$

- We can say that $(\Delta B_t)^2 \sim \Delta t$
Proof: Set $\xi_k = (\Delta B_k)^2 - \Delta t_k$. The random variables $\xi_k$ are independent and centered. Thus,

$$
E \left[ \left( \sum_{k=1}^{n} (\Delta B_k)^2 - t \right)^2 \right] = E \left[ \left( \sum_{k=1}^{n} \xi_k \right)^2 \right] = \sum_{k=1}^{n} E \left[ \xi_k^2 \right]
$$

$$
= \sum_{k=1}^{n} \left[ 3 (\Delta t_k)^2 - 2 (\Delta t_k)^2 + (\Delta t_k)^2 \right]
$$

$$
= 2 \sum_{k=1}^{n} (\Delta t_k)^2 \leq 2t |\pi| \xrightarrow{|\pi| \to 0} 0.
$$

Exercise: Using the Borel-Cantelli lemma, show that if $\{\pi^n\}$ is a sequence of partitions of $[0, t]$ such that $\sum_n |\pi^n| < \infty$, then $\sum_{k=1}^{n} (\Delta B_k)^2$ converges almost surely to $t$. 

\[\square\]
Infinite total variation

- Define

\[ V_t = \sup_{\pi} \sum_{k=1}^{n} |\Delta B_k| \]

- Then,

\[ P(V_t = \infty) = 1. \]

In fact, using the continuity of the trajectories of the Brownian motion, we have, on the set \( V < \infty \),

\[ \sum_{k=1}^{n} (\Delta B_k)^2 \leq \sup_k |\Delta B_k| \left( \sum_{k=1}^{n} |\Delta B_k| \right) \leq V \sup_k |\Delta B_k| \xrightarrow{|\pi| \to 0} 0. \]

Then, \( V < \infty \) contradicts the fact that \( \sum_{k=1}^{n} (\Delta B_k)^2 \) converges in \( L^2 \) to \( t \) as \( |\pi| \to 0 \).
**Fine properties of the trajectories**

- **Lévy’s modulus of continuity**: 
  \[
  \limsup_{\delta \downarrow 0} \sup_{s,t \in [0,1], |t-s| < \delta} \frac{|B_t - B_s|}{\sqrt{2|t-s| \log |t-s|}} = 1, \quad \text{a.s.}
  \]

- In contrast, the behavior at a single point is given by the law of iterated logarithm: 
  \[
  \limsup_{t \downarrow s} \frac{|B_t - B_s|}{\sqrt{2|t-s| \log \log |t-s|}} = 1, \quad \text{a.s.}
  \]
  for any \( s \geq 0 \).
Conditional expectation

Let $X$ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G} \subset \mathcal{F}$ a $\sigma$-algebra.

**Definition**

The conditional expectation $E(X|\mathcal{G})$ is a random variable $Y$ satisfying:

(i) $Y$ is $\mathcal{G}$-measurable.

(ii) For all $A \in \mathcal{G}$,

$$\int_A XdP = \int_A YdP.$$

- If $X \geq 0$, $E(X|\mathcal{G})$ is the density of the measure $\mu(A) = \int_A XdP$, restricted to $\mathcal{G}$, with respect to $P$.

- By the Radon-Nikodym theorem, $E(X|\mathcal{G})$ exists and it is unique almost surely.
Properties of the conditional expectation

1. **Linearity** :
   \[ E(aX + bY|\mathcal{G}) = aE(X|\mathcal{G}) + bE(Y|\mathcal{G}). \]

2. \( E(E(X|\mathcal{G})) = E(X). \)

3. If \( X \) and \( \mathcal{G} \) are independent, then \( E(X|\mathcal{G}) = E(X) \).

4. If \( X \) is \( \mathcal{G} \)-measurable, then \( E(X|\mathcal{G}) = X \).

5. If \( Y \) is bounded and \( \mathcal{G} \)-measurable, then
   \[ E(YX|\mathcal{G}) = YE(X|\mathcal{G}). \]

6. Given two \( \sigma \)-fields \( \mathcal{B} \subset \mathcal{G} \), then
   \[ E(E(X|\mathcal{B})|\mathcal{G}) = E(E(X|\mathcal{G})|\mathcal{B}) = E(X|\mathcal{B}). \]
7. Let $X$ and $Z$ be such that:
   
   (i) $Z$ is $\mathcal{G}$-measurable.
   (ii) $X$ is independent of $\mathcal{G}$.

   Suppose that $E \left( |h(X, Z)| \right) < \infty$. Then,

   $$E \left( h(X, Z) \big| \mathcal{G} \right) = E \left( h(X, z) \right) \big|_{z=Z}. $$
Markov processes

- A *filtration* \( \{ \mathcal{F}_t \subset \mathcal{F}, t \geq 0 \} \) is an increasing family of \( \sigma \)-fields.
- A process \( \{ X_t, t \geq 0 \} \) is \( \mathcal{F}_t \)-adapted if \( X_t \) is \( \mathcal{F}_t \)-measurable for all \( t \geq 0 \).

**Definition**

An adapted process \( X_t \) is a Markov process with respect to \( \mathcal{F}_t \) if for any \( s \geq 0 \), \( t > 0 \) and any \( f \in C_b(\mathbb{R}) \),

\[
E[f(X_{s+t})|\mathcal{F}_s] = E[f(X_{s+t})|X_s], \quad \text{a.s.}
\]

- This implies that \( X_t \) is also an \( \mathcal{F}_t^X \)-Markov process, where \( \mathcal{F}_t^X = \sigma \{ X_u, 0 \leq u \leq t \} \).
- The finite-dimensional marginal distributions of a Markov process are characterized by the transition probabilities

\[
p(s, x, s + t, B) = P(X_{s+t} \in B|X_s = x).
\]
Markov property of Brownian motion

**Theorem**

The Brownian motion $B_t$ is an $\mathcal{F}_t^B$-Markov process such that, for any $f \in C_b(\mathbb{R})$, $s \geq 0$ and $t > 0$,

$$E[f(B_{s+t})|\mathcal{F}_s^B] = (P_t f)(B_s),$$

where $(P_t f)(x) = \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x-y|^2}{2t}} dy$.

- $\{P_t, t \geq 0\}$ is the semigroup of operators associated with the Brownian motion:

  $$P_t \circ P_s = P_{t+s}$$

  $$P_0 = \text{Id}$$
Proof:

We have

\[ E[f(B_{s+t}) | \mathcal{F}_s^B] = E[f(B_{s+t} - B_s + B_s) | \mathcal{F}_s^B]. \]

Since \( B_{s+t} - B_s \) is independent of \( \mathcal{F}_s^B \), we obtain

\[
E[f(B_{s+t}) | \mathcal{F}_s^B] = E[f(B_{s+t} - B_s + x)]_{x=B_s} = \int_{\mathbb{R}} f(y + B_s) \frac{1}{\sqrt{2\pi t}} e^{-\frac{|y|^2}{2t}} dy
\]

\[
= \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{2\pi t}} e^{-\frac{|B_s-y|^2}{2t}} dy = (P_t f)(B_s).
\]

\[ \square \]
Multidimensional Brownian motion

- $B_t = (B^1_t, \ldots, B^d_t)$ is called a $d$-dimensional Brownian motion if its components are independent Brownian motions.

- It is a Markov process with semigroup

\[
(P_t f)(x) = \int_{\mathbb{R}^d} f(y) (2\pi t)^{-\frac{d}{2}} \exp \left( -\frac{|x - y|^2}{2t} \right) \, dy.
\]

- The transition density $p_t(x, y) = (2\pi t)^{-\frac{d}{2}} \exp \left( -\frac{|x - y|^2}{2t} \right)$ satisfies the heat equation

\[
\frac{\partial p}{\partial t} = \frac{1}{2} \Delta p, \quad t > 0,
\]

with initial condition $p_0(x, y) = \delta_x(y)$. 


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Consider a filtration \( \{\mathcal{F}_t, t \geq 0\} \) in a probability space \((\Omega, \mathcal{F}, P)\), that satisfies the following conditions:

(i) If \( A \in \mathcal{F} \) is such that \( P(A) = 0 \), then \( A \in \mathcal{F}_0 \).
(ii) The filtration is \textit{right-continuous}, that is, for every \( t \geq 0 \),

\[
\mathcal{F}_t = \bigcap_{n \geq 1} \mathcal{F}_{t + \frac{1}{n}}.
\]

**Definition**

A random variable \( T : \Omega \to [0, \infty] \) is a \textit{stopping time} with respect to a filtration \( \{\mathcal{F}_t, t \geq 0\} \) if

\[
\{ T \leq t \} \in \mathcal{F}_t, \quad \forall t \geq 0.
\]
Properties of stopping times

1. \( T \) is a stopping time if and only if \( \{ T < t \} \in \mathcal{F}_t \) for all \( t \geq 0 \).

   \textit{Proof}:
   \[
   \{ T < t \} = \bigcup_n \{ T \leq t - \frac{1}{n} \} \in \mathcal{F}_t.
   \]

   Conversely,
   \[
   \{ T \leq t \} = \bigcap_n \{ T < t + \frac{1}{n} \} \in \bigcap \mathcal{F}_{t+\frac{1}{n}} = \mathcal{F}_t. \quad \Box
   \]

2. \( S \lor T \) and \( S \land T \) are stopping times.

3. Given a stopping time \( T \),
   \[
   \mathcal{F}_T = \{ A : A \cap \{ T \leq t \} \in \mathcal{F}_t, \text{ for all } t \geq 0 \}.\]
   is a \( \sigma \)-field.

4. \( S \leq T \Rightarrow \mathcal{F}_S \subset \mathcal{F}_T \).
5. Let \( \{X_t, t \geq 0\} \) be a continuous and adapted process. The \textit{hitting time} of a set \( A \subset \mathbb{R} \) is defined by

\[
T_A = \inf\{t \geq 0 : X_t \in A\}.
\]

Then, if \( A \) is open or closed, \( T_A \) is a stopping time.

6. Let \( X_t \) be an adapted stochastic process with right-continuous paths and \( T < \infty \) a stopping time. Then the random variable

\[
X_T(\omega) = X_{T(\omega)}(\omega)
\]

is \( \mathcal{F}_T \)-measurable.
Martingales

- We assume that \( \{\mathcal{F}_t, t \geq 0\} \) is a filtration.

**Definition**

An adapted process \( M = \{M_t, t \geq 0\} \) is called a *martingale* with respect to \( \mathcal{F}_t \) if

(i) For all \( t \geq 0 \), \( E(|M_t|) < \infty \).

(ii) For each \( s \leq t \), \( E(M_t | \mathcal{F}_s) = M_s \).
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Martingales

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**Definition**

An adapted process $M = \{M_t, t \geq 0\}$ is called a *martingale* with respect to $\mathcal{F}_t$ if

(i) For all $t \geq 0$, $E(|M_t|) < \infty$.

(ii) For each $s \leq t$, $E(M_t | \mathcal{F}_s) = M_s$.

- Property (ii) can also be written as:

$$E(M_t - M_s | \mathcal{F}_s) = 0$$

- $M_t$ is a *supermartingale* (or *submartingale*) if property (ii) is replaced by $E(M_t | \mathcal{F}_s) \leq M_s$ (or $E(M_t | \mathcal{F}_s) \geq M_s$).
Basic properties

1. For any integrable random variable $X$, $\{E(X|\mathcal{F}_t), t \geq 0\}$ is a martingale.

2. If $M_t$ is a submartingale, then $t \mapsto E[M_t]$ is nondecreasing.

3. If $M_t$ is a martingale and $\varphi$ is a convex function such that $E(\varphi(M_t)) < \infty$ for all $t \geq 0$, then $\varphi(M_t)$ is a submartingale.

Proof: By Jensen’s inequality, if $s \leq t$,

$$E(\varphi(M_t)|\mathcal{F}_s) \geq \varphi(E(M_t|\mathcal{F}_s)) = \varphi(M_s). \quad \square$$

In particular, if $M_t$ is a martingale such that $E(|M_t|^p) < \infty$ for all $t \geq 0$ and for some $p \geq 1$, then $|M_t|^p$ is a submartingale.
Examples:

Let $B_t$ be a Brownian motion $\mathcal{F}_t$ the filtration generated by $B_t$:

$$\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\}.$$

Then, the processes

$$M_t^{(1)} = B_t$$
$$M_t^{(2)} = B_t^2 - t$$
$$M_t^{(3)} = \exp(aB_t - \frac{a^2 t}{2})$$

where $a \in \mathbb{R}$, are martingales.
1. $B_t$ is a martingale because

$$E(B_t - B_s | \mathcal{F}_s) = E(B_t - B_s) = 0.$$
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2. For $B_t^2 - t$, we can write, using the properties of the conditional expectation, for $s < t$

$$E(B_t^2 | \mathcal{F}_s) = E(((B_t - B_s + B_s)^2 | \mathcal{F}_s)$$

$$= E((B_t - B_s)^2 | \mathcal{F}_s) + 2E((B_t - B_s) B_s | \mathcal{F}_s)$$

$$+ E(B_s^2 | \mathcal{F}_s)$$

$$= E(B_t - B_s)^2 + 2B_s E((B_t - B_s) | \mathcal{F}_s) + B_s^2$$

$$= t - s + B_s^2.$$
1. \( B_t \) is a martingale because
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E(B_t - B_s | \mathcal{F}_s) = E(B_t - B_s) = 0.
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= E(B_t - B_s)^2 + 2B_sE((B_t - B_s) | \mathcal{F}_s) + B_s^2
= t - s + B_s^2.
\]

3. Finally, for \( \exp(aB_t - \frac{a^2t}{2}) \) we have
\[
E(e^{aB_t - \frac{a^2t}{2}} | \mathcal{F}_s) = e^{aB_s}E(e^{a(B_t - B_s) - \frac{a^2t}{2}} | \mathcal{F}_s)
= e^{aB_s}E(e^{a(B_t - B_s) - \frac{a^2t}{2}})
= e^{aB_s}e^{-\frac{a^2(t-s)}{2} - \frac{a^2t}{2}} = e^{aB_s - \frac{a^2s}{2}}.
\]
Theorem (Optional Stopping Theorem)

Suppose that $M_t$ is a continuous martingale and let $S \leq T \leq K$ two bounded stopping times. Then

$$E(M_T | \mathcal{F}_S) = M_S.$$ 

This theorem implies that $E(M_T) = E(M_S)$.

In the submartingale case we have $E(M_T | \mathcal{F}_S) \geq M_S$.

As a consequence, if $T$ is a bounded stopping time,

$$M_t \quad (\text{sub)martingale} \; \Rightarrow \; M_{t \wedge T} \quad (\text{sub)martingale}$$
Proof:

- We will show that $E(M_T) = E(M_0)$.

- Assume first that $T$ takes value in a finite set:
  \[0 \leq t_1 \leq \cdots \leq t_n \leq K.\]

  Then, by the martingale property

  \[
  E(M_T) = \sum_{i=1}^{n} E(M_{t_i} \mathbf{1}_{\{T = t_i\}}) = \sum_{i=1}^{n} E(M_{t_i} \mathbf{1}_{\{T = t_i\}}) \\
  = \sum_{i=1}^{n} E(M_{t_n} \mathbf{1}_{\{T = t_i\}}) = E(M_{t_n}) = E(M_0).
  \]

- In the general case we approximate $T$ by the following nonincreasing sequence of stopping times

  \[
  \tau_n = \sum_{k=1}^{2^n} \frac{kK}{2^n} \mathbf{1}_{\{(\frac{k-1}{2^n})K \leq T < \frac{kK}{2^n}\}}.
  \]
By continuity

\[ M_{\tau_n} \overset{a.s.}{\to} M_T. \]

To show that \( E(M_0) = E(M_{\tau_n}) \to E(M_T) \), it suffices to check that the sequence \( M_{\tau_n} \) is uniformly integrable. This follows from:

\[
E(|M_{\tau_n}| \mathbb{1}_{\{|M_{\tau_n}| \geq A\}}) = \sum_{k=1}^{2^n} E(|M_{2^n_k}| \mathbb{1}_{\{|M_{2^n_k}| \geq A, \tau_n = 2^n_k\}}) \leq \sum_{k=1}^{2^n} E(|M_k| \mathbb{1}_{\{|M_{2^n_k}| \geq A, \tau_n = 2^n_k\}}) \leq E(|M_k| \mathbb{1}_{\{|M_{\tau_n}| \geq A\}}) \leq E(|M_k| \mathbb{1}_{\{\sup_{0 \leq s \leq k} |M_s| \geq A\}}),
\]

which converges to zero as \( A \uparrow \infty \), uniformly in \( n \). \( \square \)
Doob’s maximal inequalities

**Theorem**

Let \( \{M_t, t \in [0, T]\} \) be a continuous martingale such that \( E(|M_T|^p) < \infty \) for some \( p \geq 1 \). Then, for all \( \lambda > 0 \) we have

\[
P \left( \sup_{0 \leq t \leq T} |M_t| > \lambda \right) \leq \frac{1}{\lambda^p} E(|M_T|^p).
\]  

(1)

If \( p > 1 \), then

\[
E \left( \sup_{0 \leq t \leq T} |M_t|^p \right) \leq \left( \frac{p}{p - 1} \right)^p E(|M_T|^p).
\]  

(2)
Proof of (1):

- Set
  \[ \tau = \inf\{s \geq 0 : |M_s| \geq \lambda\} \wedge T. \]
  Because \( \tau \) is a bounded stopping time and \( |M_t|^p \) is a submartingale,
  \[ E(|M_\tau|^p) \leq E(|M_T|^p). \]

- From the definition of \( \tau \),
  \[ |M_\tau|^p \geq 1_{\{\sup_{0 \leq t \leq \tau} |M_t| \geq \lambda\}} \lambda^p + 1_{\{\sup_{0 \leq t \leq \tau} |M_t| < \lambda\}} |M_T|^p. \]
  Therefore,
  \[ P\left( \sup_{0 \leq t \leq T} |M_t| > \lambda \right) \leq \frac{1}{\lambda^p} E(|M_\tau|^p) \leq \frac{1}{\lambda^p} E(|M_T|^p). \]
Let $B_t$ be a Brownian motion. Fix $a \in \mathbb{R}$ and consider the hitting time

$$
\tau_a = \inf\{t \geq 0 : B_t = a\}
$$

**Proposition**

If $a < 0 < b$, then

$$
P(\tau_a < \tau_b) = \frac{b}{b-a}.
$$

**Proof :** By the optional stopping theorem

$$
E(B_{t \land \tau_a}) = E(B_0) = 0.
$$

Letting $t \to \infty$ and using the dominated convergence theorem, it follows that

$$
0 = aP(\tau_a < \tau_b) + b(1 - P(\tau_a < \tau_b)).
$$
Proposition

Let \( T = \inf\{t \geq 0 : B_t \notin (a, b)\} \), where \( a < 0 < b \). Then

\[
E(T) = -ab.
\]

Proof: Using that \( B_t^2 - t \) is a martingale, we get

\[
E(B_{T \wedge t}^2) = E(T \wedge t).
\]

Therefore,

\[
E(T) = \lim_{t \to \infty} E(B_{T \wedge t}^2) = E(B_T^2) = -ab.
\]
Proposition

Fix $a > 0$. The hitting time

$$\tau_a = \inf\{t \geq 0 : B_t = a\},$$

satisfies

$$E [\exp (-\alpha \tau_a)] = e^{-\sqrt{2\alpha} a}. \quad \alpha > 0 \quad (3)$$
**Proof:**

- For any $\lambda > 0$, the process $M_t = e^{\lambda B_t - \frac{\lambda^2 t}{2}}$ is a martingale such that
  $$E(M_t) = E(M_0) = 1.$$

- By the optional stopping theorem we obtain, for all $N \geq 1$.
  $$E(M_{\tau_a \wedge N}) = 1.$$

- Notice that $M_{\tau_a \wedge N} = \exp \left( \lambda B_{\tau_a \wedge N} - \frac{\lambda^2 (\tau_a \wedge N)}{2} \right) \leq e^{\alpha \lambda}$. So, by the dominated convergence theorem we obtain
  $$E(M_{\tau_a}) = 1,$$
  that is,
  $$E \left( \exp \left( - \frac{\lambda^2 \tau_a}{2} \right) \right) = e^{-\lambda a}.$$

With the change of variables $\frac{\lambda^2}{2} = \alpha$, we get
$$E \left( \exp \left( - \alpha \tau_a \right) \right) = e^{-\sqrt{2\alpha}a}.$$

(4)
The expectation of $\tau_a$ can be obtained by computing the derivative of (4) with respect to the variable $\alpha$:

$$E\left( \tau_a \exp (-\alpha \tau_a) \right) = \frac{ae^{-\sqrt{2\alpha}a}}{\sqrt{2\alpha}},$$

and letting $\alpha \downarrow 0$ we obtain $E(\tau_a) = +\infty$.

We can compute the density function of $\tau_a$:

$$f_{\tau_a}(s) = \frac{a}{\sqrt{2\pi}} s^{-\frac{3}{2}} e^{-a^2/2s}, \quad s \geq 0.$$
Theorem

Let $B$ be a Brownian motion and let $T$ be a finite stopping time with respect to the filtration $\mathcal{F}_t^B$ generated by $B$. Then the process

$$\{B_{T+t} - B_T, t \geq 0\}$$

is a Brownian motion independent of $B_T$.

As a consequence, for any $f \in C_b(\mathbb{R})$ and any finite stopping time $T$ for the filtration $\mathcal{F}_t^B$, we have

$$E[f(B_{T+t})|\mathcal{F}_T^B] = (P_t f)(B_T),$$

where $P_t$ is the semigroup associated with the Brownian motion $B$. 
Proof:

Consider the process $\tilde{B}_t = B_{T+t} - B_T$ and suppose first that $T$ is bounded. Let $\lambda \in \mathbb{R}$ and $0 \leq s \leq t$. Applying the optional stopping theorem to the martingale

$$\exp\left( i\lambda \tilde{B}_t + \frac{\lambda^2 t}{2} \right),$$

yields

$$E \left[ e^{i\lambda B_{T+t} + \frac{\lambda^2 (T+t)}{2}} | \mathcal{F}_{T+s} \right] = e^{i\lambda B_{T+s} + \frac{\lambda^2 (T+s)}{2}}.$$

Therefore,

$$E \left[ e^{i\lambda (B_{T+t} - B_{T+s})} | \mathcal{F}_{T+s} \right] = e^{-\frac{\lambda^2}{2} (t-s)}.$$

This implies that the increments of $\tilde{B}$ are independent, stationary and normally distributed.

If $T$ is not bounded, then we can consider the stopping time $T \wedge N$ and let $N \to \infty$. 

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Reflection principle

Theorem

Let $M_t = \sup_{0 \leq s \leq t} B_s$. Then

$$P(M_t \geq a) = 2P(B_t > a) = 2 \frac{1}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{x^2}{2}} dx.$$
Proof:

We have

\[ P(B_t \geq a) = P(B_t \geq a, M_t \geq a) = P(B_t \geq a | M_t \geq a) P(M_t \geq a) = P(B_t \geq a | \tau_a \leq t) P(M_t \geq a). \]

We know that \( \{B_{\tau_a + s} - a, s \geq 0\} \) is a Brownian motion independent of \( \mathcal{F}_{\tau_a} \). Therefore,

\[ P(B_t \geq a | \tau_a \leq t) = E[P(B_{\tau_a + (t-\tau_a)} - a \geq 0 | \mathcal{F}_{\tau_a}) | \tau_a \leq t] = \frac{1}{2}. \]
Brownian filtration

Define

$$\mathcal{F}_t^B = \sigma \{ B_s, 0 \leq s \leq t \}.$$ 

Denote by $\mathcal{N}$ the family of sets in $\mathcal{F}$ of probability zero (null sets).

**Proposition**

The filtration

$$\mathcal{F}_t = \sigma \left\{ \mathcal{F}_t^B, \mathcal{N} \right\}.$$ 

is right-continuous. Therefore, it satisfies conditions (i) and (ii).