Theoretical Tutorial Session 2

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Outline

- Itô's formula
- Martingale representation theorem
- Stochastic differential equations
1. Using Itô’s formula to show that \( M_t = B_t^3 - 3 \int_0^t B_s \, ds \) is a martingale.

**Proof:** Let \( f(x) = x^3 \in C^2(\mathbb{R}) \). Then

\[ f'(x) = 3x^2, \text{ and } f''(x) = 6x. \]

Applying Itô’s formula to \( f(B_t) \) we have

\[
 f(B_t) = f(B_0) + \int_0^t f'(B_s) \, dB_s + \frac{1}{2} \int_0^t f''(B_s) \, ds,
\]

that is,

\[
 B_t^3 = 3 \int_0^t B_s^2 \, dB_s + 3 \int_0^t B_s \, ds. \tag{1}
\]

Then

\[
 M_t = B_t^3 - 3 \int_0^t B_s \, ds = 3 \int_0^t B_s^2 \, dB_s.
\]
Since

\[ E \left( \int_0^t (B_s^2)^2 ds \right) = E \left( \int_0^t B_s^4 ds \right) = \int_0^t 3s^2 ds < \infty \]

for all \( t \geq 0 \), by the basic property of indefinite Itô integral, we can show that

\[ M_t = B_t^3 - 3 \int_0^t B_s ds = 3 \int_0^t B_s^2 dB_s \]

is a martingale.
2. Use Itô’s formula to show that

\[ t B_t = \int_0^t B_s ds + \int_0^t s dB_s. \] (2)

**Proof:** Let \( f(t, x) = tx \). Then \( f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}) \) and

\[
\frac{\partial f}{\partial t}(t, x) = x,
\]
\[
\frac{\partial f}{\partial x}(t, x) = t,
\]
\[
\frac{\partial^2 f}{\partial x^2}(t, x) = 0.
\]
Applying Itô’s formula

\[ f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s) \, ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s) \, dB_s 
+ \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s) \, ds, \]

we obtain

\[ tB_t = \int_0^t B_s \, ds + \int_0^t s \, dB_s. \]

Note that the above equation give us an integration by parts formula.
3. Check if the process $X_t = B_t^3 - 3tB_t$ is a martingale.

Solution: Using (1) and (2), we can write

$$X_t = B_t^3 - 3tB_t$$

$$= 3 \int_0^t B_s^2 dB_s + 3 \int_0^t B_s ds - 3 \left( \int_0^t B_s ds + \int_0^t s dB_s \right)$$

$$= \int_0^t (3B_s^2 - 3s) dB_s.$$

We can also show that

$$E \left( \int_0^t (3B_s^2 - 3s)^2 ds \right) < \infty, \ \forall t \geq 0.$$  

Therefore, the process $X_t = B_t^3 - 3tB_t$ is a martingale.
4. Find the stochastic integral representation on the time interval \([0, T]\) of the square integrable random variable \(B_T^3\).

**Solution:** Using (1) and (2) with \(t = T\), we have

\[
B_T^3 = 3 \int_0^T B_s^2 dB_s + 3 \int_0^T B_s ds \\
= 3 \int_0^T B_s^2 dB_s + 3 \left( TB_T - \int_0^T s dB_s \right) \\
= 3 \int_0^T B_s^2 dB_s + 3 \left( T \int_0^T 1 dB_s - \int_0^T s dB_s \right) \\
= \int_0^T \left( 3B_s^2 + 3T - 3s \right) dB_s.
\]

The above is the integral representation for \(B_T^3\) since the process \(\{2B_s + 3T - 3s, \ s \in [0, T]\}\) is in \(L^2(P)\) and \(E(B_T^3) = 0\).
5. Verify that the following processes are martingales:

(a) \( X_t = t^2 B_t - 2 \int_0^t sB_s ds \)

(b) \( X_t = e^{t/2} \cos B_t \)

(c) \( X_t = e^{t/2} \sin B_t \)

(d) \( X_t = B_1(t) B_2(t) \), where \( B_1 \) and \( B_2 \) are two independent Brownian motion.

Solution 5(a): Let \( f(t, x) = t^2 x \). Then \( f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}) \) and

\[
\frac{\partial f}{\partial t}(t, x) = 2tx, \\
\frac{\partial f}{\partial x}(t, x) = t^2, \\
\frac{\partial^2 f}{\partial x^2}(t, x) = 0.
\]
Applying Itô’s formula

\[ f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s \]

\[ + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)ds, \]

we get

\[ t^2 B_t = \int_0^t 2sB_s ds + \int_0^t s^2 dB_s. \]

Hence, the process

\[ X_t = t^2 B_t - 2 \int_0^t sB_s ds = \int_0^t s^2 dB_s \]

is a martingale.
Solution 5(b): Let \( f(t, x) = e^{t/2} \cos x \). Then \( f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}) \) and

\[
\frac{\partial f}{\partial t}(t, x) = \frac{1}{2} f(t, x),
\]

\[
\frac{\partial f}{\partial x}(t, x) = -e^{t/2} \sin x,
\]

\[
\frac{\partial^2 f}{\partial x^2}(t, x) = -f(t, x).
\]

Note that

\[
\frac{\partial f}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, x) = 0, \text{ and } f(0, B_0) = 1.
\]

Then we apply Itô’s formula and show that the process

\[
X_t = e^{t/2} \cos B_t = 1 - \int_0^t e^{s/2} \sin B_s dB_s
\]

is a martingale since \( E(\int_0^t e^{s} \sin B_s^2 ds) \leq \int_0^t e^{s} ds < \infty \) for all \( t \geq 0 \).
Solution 5(c): Let \( f(t, x) = e^{t/2} \sin x \). Then \( f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}) \) and

\[
\frac{\partial f}{\partial t}(t, x) = \frac{1}{2} f(t, x),
\]

\[
\frac{\partial f}{\partial x}(t, x) = e^{t/2} \cos x,
\]

\[
\frac{\partial^2 f}{\partial x^2}(t, x) = -f(t, x).
\]

Note that

\[
\frac{\partial f}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, x) = 0, \quad \text{and} \quad f(0, B_0) = 0.
\]

Then we apply Itô’s formula and show that the process

\[
X_t = e^{t/2} \sin B_t = \int_0^t e^{s/2} \cos B_s dB_s
\tag{3}
\]

is a martingale since \( E(\int_0^t e^s \cos B_s^2 ds) \leq \int_0^t e^s ds < \infty \) for all \( t \geq 0 \).
Solution 5(d): For this exercise, we need to apply Itô’s formula in multidimensional case. Let \( f(x_1, x_2) = x_1 x_2 \). Then \( f \in C^2(\mathbb{R}^2) \) and

\[
\begin{align*}
\frac{\partial f}{\partial x_1}(x_1, x_2) &= x_2 \\
\frac{\partial f}{\partial x_2}(x_1, x_2) &= x_1 \\
\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) &= 0 \\
\frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) &= 0 \\
\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) &= 1
\end{align*}
\]
Applying multidimensional Itô’s formula, one can obtain

\[ f(B_1(t)B_2(t)) = f(B_1(0)B_2(0)) + \int_0^t \frac{\partial f}{\partial x_1}(B_1(s), B_2(s))dB_1(s) \]

\[ + \int_0^t \frac{\partial f}{\partial x_2}(B_1(s), B_2(s))dB_2(s) \]

\[ + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x_1^2}(B_1(s), B_2(s))ds \]

\[ + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x_2^2}(B_1(s), B_2(s))ds \]

\[ + \int_0^t \frac{\partial^2 f}{\partial x_1 \partial x_2}(B_1(s), B_2(s))dB_1(s)dB_2(s), \]

then noticing that \( dB_1 dB_2 = 0 \), we can show that the process

\[ X_t = B_1(t)B_2(t) = \int_0^t B_2(s)dB_1(s) + \int_0^t B_1(s)dB_2(s) \]

is a martingale, since

\[ E\left( \int_0^t B_1(s)^2 ds \right) = E\left( \int_0^t B_2(s)^2 ds \right) = \int_0^t sds < \infty, \ \forall t \geq 0. \]
6. If \( f(t, x) = e^{ax - \frac{a^2}{2} t} \) and \( Y_t = f(t, B_t) = e^{aB_t - \frac{a^2}{2} t} \) where \( a \) is a constant, then prove that \( Y \) satisfies the following linear SDE:

\[
Y_t = 1 + a \int_0^t Y_s dB_s. \tag{4}
\]

**Proof:** Note that \( f(t, x) \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}) \) and

\[
\frac{\partial f}{\partial t}(t, x) = -\frac{a^2}{2} f(t, x),
\]
\[
\frac{\partial f}{\partial x}(t, x) = af(t, x),
\]
\[
\frac{\partial^2 f}{\partial x^2}(t, x) = a^2 f(t, x).
\]

Note also that

\[
\frac{\partial f}{\partial t}(t, x) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, x) = 0.
\]
Applying Itô’s formula, we have

\[
f(t, B_t) = f(0, B_0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s \\
+ \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)ds
\]

\[
= 1 + \int_0^t \frac{\partial f}{\partial x}(s, B_s)dB_s,
\]

that is

\[
Y_t = 1 + a \int_0^t Y_sdB_s.
\]
Remark: i). Note that

\[
E \left( \int_0^t |Y_s|^2 ds \right) = E \left( \int_0^t e^{2aB_s-a^2s} ds \right) \\
= \int_0^t e^{a^2s} E \left( e^{2aB_s-(2a)^2} \right) ds \\
= \int_0^t e^{a^2s} ds < \infty,
\]

for all \( t \geq 0 \). Hence, the Itô integral \( \int_0^t Y_s dB_s \) is well-defined.

ii). The solution of the stochastic differential equation

\[
dY_t = aY_t dB_t, \quad Y_0 = 1
\]

is not \( Y_t = e^{aB_t} \), but \( Y_t = e^{aB_t - \frac{a^2}{2} t} \).
7. Find the stochastic integral representation on the time interval \([0, T]\) of the following square integrable random variables:

(a) \( F = B_T \)
(b) \( F = B_T^2 \)
(c) \( F = e^{B_T} \)
(d) \( F = \sin B_T \)
(e) \( F = \int_0^T B_t \, dt \)
(f) \( F = \int_0^T tB_t^2 \, dt \)

Solution 7(a): Since \( E(B_T) = 0 \), the stochastic integral representation for \( B_T \) is

\[
B_T = \int_0^T 1 \, dB_t = E(B_T) + \int_0^T 1 \, dB_t.
\]
Solution 7(b): Let \( f(x) = x^2 \). Then \( f \in C^2(\mathbb{R}) \) and \( f'(x) = 2x \) and \( f''(x) = 2 \). Using Itô’s formula and \( E(B_T^2) = T \), we have

\[
B_T^2 = B_0^2 + \int_0^T 2B_s dB_s + \frac{1}{2} \int_0^T 2dt
\]

\[
= \int_0^T 2B_s dB_s + T
\]

\[
= E(B_T^2) + \int_0^T 2B_s dB_s.
\]
Solution 7(c): We can calculate that

\[ E(e^{B_T}) = e^{\frac{T}{2}} \]

In fact, we can NOT apply Itô’s formula directly to get the stochastic integral representation, since if we choose \( f(x) = e^x \) and apply Itô’s formula to \( f(B_T) \), then we get

\[
e^{B_T} = e^{B_0} + \int_0^T e^{B_t} dB_t + \frac{1}{2} \int_0^T e^{B_t} dt
\]

\[ = 1 + \int_0^T e^{B_t} dB_t + \frac{1}{2} \int_0^T e^{B_t} dt. \]

We can not get rid of the integral with respect to \( dt \).

**Question:** How can we get its stochastic integral representation?
In order to obtain the stochastic integral representation for $e^{B_T}$, we will need the result (4) in Exercise 6 with $a = 1$ and $t = T$:

$$e^{B_T - \frac{T}{2}} = 1 + \int_0^T e^{B_t - \frac{t}{2}} dB_t.$$

Multiplying $e^{\frac{T}{2}}$ on both sides of the above equation, we obtain the following stochastic integral representation

$$e^{B_T} = e^{\frac{T}{2}} + e^{\frac{T}{2}} \int_0^T e^{B_t - \frac{t}{2}} dB_t$$

$$= E(e^{B_T}) + \int_0^T e^{B_t + \frac{T-t}{2}} dB_t.$$
Solution 7(d): Note that

\[ E(\sin B_T) = 0. \]

Since \( \sin x \) and \( e^x \) are closely related in

\[ e^{ix} = \cos x + i \sin x, \]

we can foresee the same problem if we apply Itô’s formula directly to \( f(B_T) = \sin B_T \).

Instead, we make use of (3) and obtain

\[
\sin B_T = e^{-\frac{t}{2}} \int_0^T e^{\frac{t}{2}} \cos B_t dB_t
\]

\[ = E(\sin B_T) + \int_0^T e^{-\frac{T-t}{2}} \cos B_t dB_t. \]
Solution 7(e): We have

\[ E \left( \int_0^T B_t \, dt \right) = 0. \]

Using (2) with \( t = T \) we get

\[ \int_0^T B_t \, dt = TB_T - \int_0^T t \, dB_t \]

\[ = T \int_0^T 1 \, dB_t - \int_0^T t \, dB_t \]

\[ = \int_0^T (T - t) \, dB_t \]

\[ = E \left( \int_0^T B_t \, dt \right) + \int_0^T (T - t) \, dB_t. \]
Solution 7(f): Note that

\[ E \left( \int_0^T tB_t^2 \, dt \right) = \int_0^T tE(B_t^2) \, dt = \int_0^T t^2 \, dt = \frac{T^3}{3}. \]

From Part 7(b), we know

\[ B_t^2 = 2 \int_0^t B_s dB_s + t, \quad \forall t \geq 0. \]

Using the above equation and then changing the order of the integrals we have

\[ \int_0^T tB_t^2 \, dt = \int_0^T t \left( 2 \int_0^t B_s dB_s + t \right) \, dt 
= \int_0^T t^2 \, dt + 2 \int_0^T \int_0^t tB_s dB_s dt 
= \frac{T^3}{3} + 2 \int_0^T B_s \left( \int_s^T t dt \right) dB_s 
= E \left( \int_0^T tB_t^2 \, dt \right) + \int_0^T (T^2 - s^2) B_s dB_s. \]
8. Consider an $n$-dimensional Brownian motion $B(t) = (B_1(t), B_2(t), \ldots, B_n(t))$ and constants $\alpha_i$, $i = 1, \ldots, n$. 

Solve the following SDE:

$$dX_t = rX_t dt + X_t \sum_{i=1}^{n} \alpha_i dB_i(t), \quad X_0 = x,$$

where $x \in \mathbb{R}$.

**Solution:** The coefficients in this SDE satisfy the Lipschitz and linear growth conditions, so there exists a unique solution.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then the above SDE becomes an ODE

$$dX_t = rX_t dt, \quad X_0 = x.$$

and its unique solution is $X_t = xe^{rt}$. 
If $\sum_{i=1}^{n} \alpha_i^2 \neq 0$, then by using the standard method mentioned in my first tutorial session we can show that the process

$$\tilde{B}_t = \frac{1}{\sqrt{\sum_{i=1}^{n} \alpha_i^2}} \sum_{i=1}^{n} \alpha_i B_i(t)$$

is a Brownian motion.

Let $Y_t = e^{-rt}$. Then $Y$ satisfies $dY_t = -rY_t \, dt$. Applying multidimensional Itô’s formula to $f(x, y) = xy$ we have

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t$$

$$= X_t dY_t + Y_t dX_t$$

$$= X_t (-rY_t) \, dt + Y_t (rX_t \, dt + X_t \sum_{i=1}^{n} \alpha_i dB_i(t))$$

$$= X_t Y_t \sum_{i=1}^{n} \alpha_i dB_i(t)$$

$$= \sqrt{\sum_{i=1}^{n} \alpha_i^2} (X_t Y_t) \, d\tilde{B}_t. \quad (5)$$
Thus, $X_t Y_t$ satisfies the linear SDE (5). Note also that $X_0 Y_0 = x$. Then the solution to (5) is

$$X_t Y_t = x \exp \left\{ \sqrt{\sum_{i=1}^{n} \alpha_i^2 \tilde{B}_t} - \frac{t \sum_{i=1}^{n} \alpha_i^2}{2} \right\}. $$

Therefore,

$$X_t = Y_t^{-1} x \exp \left\{ \sqrt{\sum_{i=1}^{n} \alpha_i^2 \tilde{B}_t} - \frac{t \sum_{i=1}^{n} \alpha_i^2}{2} \right\} = x \exp \left\{ rt + \sqrt{\sum_{i=1}^{n} \alpha_i^2 \tilde{B}_t} - \frac{t \sum_{i=1}^{n} \alpha_i^2}{2} \right\} = x \exp \left\{ \sum_{i=1}^{n} \alpha_i B_i(t) + \left( r - \frac{\sum_{i=1}^{n} \alpha_i^2}{2} \right) t \right\}. $$
9. Solve the following stochastic differential equations

\[ dX_t = \frac{1}{X_t} dt + \alpha X_t dB_t, \quad X_0 = x > 0. \]

For which values of the parameter \( \alpha \) the solution explodes?

**Solution:** Let \( Y_t = e^{-\alpha B_t - \frac{\alpha^2 t}{2}} \). Then \( Y_t \) satisfies the following SDE:

\[ dY_t = -\alpha Y_t dB_t, \quad Y_0 = 1. \]

Using Itô’s formula, we have

\[
d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t
\]

\[
= X_t (-\alpha Y_t dB_t) + Y_t \left( \frac{1}{X_t} dt + \alpha X_t dB_t \right) - \alpha^2 X_t Y_t dt
\]

\[
= \frac{Y_t}{X_t} dt - \alpha^2 X_t Y_t dt,
\]

which implies
\[
2(X_t Y_t) d(X_t Y_t) = 2 Y_t^2 dt - 2 \alpha^2 (X_t Y_t)^2 dt.
\]

Then for each fixed \( \omega \), \((X_t(\omega) Y_t(\omega))\) solves the following linear ODE:

\[
\dot{y} = 2 Y_t^2(\omega) - 2 \alpha^2 y, \quad y(0) = x^2,
\]

whose solution is given by

\[
y(t) = e^{-2 \alpha^2 t} \left( x^2 + 2 \int_0^t e^{2 \alpha^2 s} Y_s^2(\omega) \right) ds.
\]

Then

\[
X_t = Y_t^{-1} e^{-\alpha^2 t} \sqrt{x^2 + 2 \int_0^t e^{2 \alpha^2 s} Y_s^2 ds}.
\]

Since the trajectories of the process \( Y \) are continuous on \([0, \infty)\) almost surely, the above integral is well defined for all \( t \geq 0 \), and hence \( X_t \) will not explode for any parameter \( \alpha \).
10. Solve the following stochastic differential equations

\[ dX_t = X_t^\gamma \, dt + \alpha X_t \, dB_t, \quad X_0 = x > 0. \]

For which values of the parameters \( \gamma, \alpha \) does the solution explode?

**Solution:** If \( \alpha = 0 \), then the differential equation is an ODE

\[ \frac{d}{dt} X = X^\gamma, \quad X_0 = x > 0. \]

This is a separable equation and we know that the solution explodes when \( \gamma > 1 \).

If \( \alpha \neq 0 \) and \( \gamma = 1 \), then this is a linear SDE and its solution is given by

\[ X_t = e^{\alpha B_t + \left(1 - \frac{\alpha^2}{2}\right)t}, \quad \forall t \geq 0. \]
If \( \alpha \neq 0 \) and \( \gamma \neq 1 \), then we will use very similar steps as in Problem 9 to obtain the solution. Let \( Y_t = e^{-\alpha B_t - \frac{\alpha^2 t}{2}} \). Then \( Y_t \) satisfies the following SDE:

\[
dY_t = -\alpha Y_t dB_t, \quad Y_0 = 1,
\]

Using Itô’s formula, we have

\[
d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t dY_t \\
= X_t (-\alpha Y_t dB_t) + Y_t \left( X_t^\gamma dt + \alpha X_t dB_t \right) - \alpha^2 X_t Y_t dt \\
= Y_t X_t^\gamma dt - \alpha^2 X_t Y_t dt,
\]

which implies for each fixed \( \omega \in \Omega \), \( y(t) = X_t(\omega) Y_t(\omega) \) satisfies the following nonlinear ODE:

\[
\dot{y} = Y_t(\omega)^{1-\gamma} y^\gamma - \alpha^2 y, \quad y(0) = x,
\]
or equivalently,

\[ \dot{y} + \alpha^2 y = Y_t(\omega)^{1-\gamma} y^{\gamma}, \quad y(0) = x. \]

Multiplying the above equation by \( e^{\alpha^2 t} \) and denoting \( z = e^{\alpha^2 t} y \), we obtain

\[ \dot{z} = \left( Y_t(\omega)e^{\alpha^2 t} \right)^{1-\gamma} z^{\gamma}, \quad z(0) = x. \]

We can separate the variables to solve this ODE as follows:

\[
\frac{\dot{z}}{z^{\gamma}} = \left( Y_t(\omega)e^{\alpha^2 t} \right)^{1-\gamma} = e^{-\alpha(1-\gamma)B_t + \frac{\alpha^2(1-\gamma)t}{2}}, \quad z(0) = x. \tag{6}
\]
Note that

\[ z(t) = X_t(\omega) e^{-\alpha B_t(\omega) + \frac{1}{2} \alpha^2 t} \]

and \( e^{-\alpha B_t(\omega) + \frac{1}{2} \alpha^2 t} \) is continuous in \( t \). Then, \( X_t(\omega) \) explodes as \( t \uparrow T(\omega) \) if and only if \( z(t) \) explodes when \( t \uparrow T(\omega) \).

Suppose that \( X_t(\omega) \) explodes as \( t \uparrow T(\omega) \) for some \( T(\omega) < \infty \). Then integrating (6) on both sides, we should get

\[ \int_\chi^\infty \frac{dz}{z^\gamma} = \int_0^T e^{-\alpha (1-\gamma) B_t + \frac{\alpha^2 (1-\gamma) t}{2}} dt < \infty, \]

(7)

and hence, explosion might occur only if \( \gamma > 1 \).
For $\gamma > 1$, then
\[
\int_{x}^{\infty} \frac{dz}{z^{\gamma}} = \frac{x^{1-\gamma}}{\gamma - 1}.
\]

Thus, if $X_t(\omega)$ explodes as $t \uparrow T(\omega)$ for some $T(\omega) < \infty$, the following equation holds
\[
\int_{0}^{T} e^{-\alpha(1-\gamma)B_t + \frac{\alpha^2(1-\gamma)t}{2}} dt = \frac{x^{1-\gamma}}{\gamma - 1}.
\]

Note also that for each $t \geq 0$ we have
\[
E \left( \int_{0}^{t} e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)s}{2}} ds \right)
= \int_{0}^{t} e^{\frac{\alpha^2(1-\gamma)^2s}{2} + \frac{\alpha^2(1-\gamma)s}{2}} ds
= \int_{0}^{t} e^{\frac{\alpha^2(1-\gamma)(2-\gamma)s}{2}} ds
= \begin{cases} 
  t, & \text{if } \gamma = 2, \\
  \frac{2}{\alpha^2(1-\gamma)(2-\gamma)} \left( e^{\frac{\alpha^2(1-\gamma)(2-\gamma)t}{2}} - 1 \right), & \text{if } \gamma \neq 2.
\end{cases}
\]
So, for $\alpha \neq 0$ and $\gamma \geq 2$, we have

$$
\lim_{t \to \infty} E \left( \int_0^t e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)s}{2}} ds \right) = \infty.
$$

(9)

Define

$$
\tau = \inf \left\{ t \geq 0, \int_0^t e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)s}{2}} ds = \frac{x^{1-\gamma}}{\gamma - 1} \right\}.
$$

That is, $z$ (or equivalently, $X$) explodes at $\tau$.

Then $\tau$ is a stopping time, and moreover, from (9), we get

$$
P(\tau < \infty) > 0,
$$

that is, $X$ explodes on $\{\tau < \infty\}$.
For $\alpha \neq 0$ and $1 < \gamma < 2$, we get from (8)

$$\lim_{t \to \infty} E \left( \int_0^t e^{-\alpha(1-\gamma)B_s + \frac{\alpha^2(1-\gamma)s}{2}} ds \right) = \frac{2}{\alpha^2(\gamma - 1)(2 - \gamma)}.$$ 

If $\alpha$ and $\gamma$ satisfy

$$\frac{2}{\alpha^2(\gamma - 1)(2 - \gamma)} > \frac{x^{1-\gamma}}{\gamma - 1},$$

then we also get

$$P(\tau < \infty) > 0,$$

that is, $X$ explodes on $\{ \tau < \infty \}$ in this case.