Bridge representation and small time approximation of the joint density

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Outline

- Preliminaries
- Bridge representations for transition density
- Probabilistic derivation of the heat kernel expansion
- Extensions to processes driven by fractional Brownian motions: mixed Brownian and fractional Brownian motions with drift
Let $\mathbf{X} = (X_1, \cdots, X_n)'$ and $\mathbf{Y} = (Y_1, \cdots, Y_m)'$ be joint Gaussian random vectors and $\mathbf{Z} = (X_1, \cdots, X_n, Y_1, \cdots, Y_m)'$. Denote the expectations of $\mathbf{X}$, $\mathbf{Y}$ and the covariance matrix for $\mathbf{Z}$ by

$$\mathbb{E}[\mathbf{X}] = \mu_\mathbf{X}, \quad \mathbb{E}[\mathbf{Y}] = \mu_\mathbf{Y}$$

and

$$\Sigma = \text{Cov} \left[ \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \right] = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}$$
Conditional expectation of Gaussian random vectors

Lemma

Suppose that the covariance matrix $\Sigma$ is positive definite. Then, the conditional distribution of $X$ given that $Y = y$ is $n$-dimensional Gaussian with expectation and covariance matrix respectively,

$$\mathbb{E}[X|Y = y] = \mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(y - \mu_Y),$$

$$\text{Cov}[X|Y = y] = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}.$$  

Moreover, the Gaussian vector $X$ has the following decomposition

$$X = \mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(Y - \mu_Y) + V,$$

where the random vector $V$ is $n$-dimensional Gaussian with zero expectation and the following covariance matrix

$$\text{Cov}[V] = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}.$$
Gaussian bridge

Let $X$ be a $\mathbb{R}^d$-valued continuous Gaussian process with $X_0 = x$ and $\mathbb{E}[X_t] = x$ for all $t \in [0, T]$. For any given $y \in \mathbb{R}^d$, then the Gaussian process defined by

$$X_{t}^{x, y} = X_t + \Sigma(t; T) \Sigma(T)^{-1}(y - X_T)$$

is a Gaussian bridge of $X$ in the sense that $X_0^{x, y} = x$ and $X_T^{x, y} = y$, where

$$\Sigma(t; T) = \text{Cov}(X_t, X_T)$$

and

$$\Sigma(T) = \Sigma(T; T) = \text{Cov}(X_T, X_T).$$

Moreover, we have

$$\mathbb{P}\text{-Law} (\{X_{t}^{x, y}\}_{t \in [0, T]}) = \mathbb{P}\text{-Law} (\{X_t\}_{t \in [0, T]} | X_T = y).$$
Consider the diffusion process

$$\begin{cases}
dS_t = a(S_t, t)dB_t + b(S_t, t)dt, & t \in [0, T], \\
S_0 = s_0,
\end{cases}$$

with $a(\xi, t) \geq \epsilon > 0$ for all $(\xi, t)$. 
A Brownian bridge representation for the transition density $p$ can be derived, which will lead to the derivation of the heat kernel expansion for the transition density.

- One dimensional nondegenerate diffusion is easier to deal with because we can always unitize the diffusion coefficient by applying the Lamperti transformation.
- Such transformations generally do not exist in higher dimensions due to geometry obstructions.
Lamperti transformation

Let

$$\varphi(x, t) = \int_{s_0}^{x} \frac{d\xi}{a(\xi, t)}.$$ 

By Itô’s formula, $X_t = \varphi(S_t, t)$ satisfies the SDE

$$dX_t = dB_t + h(X_t, t)dt$$

where $h(x, t) := \varphi_t + \frac{b}{a} - \frac{a^2}{2}$. Therefore, $X_t$ is a Brownian motion with a drift. Define

$$\frac{d\tilde{P}}{dP} = e^{-\int_0^T h(X_s, s)dB_s - \frac{1}{2} \int_0^T h^2(X_s, s)ds}$$

$$= e^{-\int_0^T h(X_s, s)dX_s + \frac{1}{2} \int_0^T h^2(X_s, s)ds}.$$ 

By Girsanov’s theorem, under the probability measure $\tilde{P}$, $X_t$ is a Brownian motion.
Girsanov transformation

Given any bounded measurable function $f$, we have

$$
\mathbb{E}[f(X_T)] = \mathbb{E} \left[ f(X_T) \frac{d\mathbb{P}}{d\mathbb{P}'} \right] 
= \mathbb{E} \left[ f(X_T) e^{\int_0^T h(X_s, s) dX_s - \frac{1}{2} \int_0^T h^2(X_s, s) ds} \right].
$$

In other words, let $p$ be the transition density of $X$,

$$
\int f(y) p(T, y|0, x) dy
= \int f(y) \mathbb{E}_y \left[ e^{\int_0^T h(X_s, s) dX_s - \frac{1}{2} \int_0^T h^2(X_s, s) ds} \right] \frac{e^{-(y-x)^2}}{\sqrt{2\pi T}} dy,
$$
where $\tilde{E}_y[\cdot] = \tilde{E}[\cdot|X_T = y]$, and hence

$$p(T, y|0, x) = \tilde{E}_y \left[ e^{\int_0^T h(X_s, s) dX_s} - \frac{1}{2} \int_0^T h^2(X_s, s) ds \right] e^{-\frac{(y-x)^2}{2T}} \frac{1}{\sqrt{2\pi T}}.$$

Furthermore, by applying Ito’s formula, we rewrite the stochastic integral term as

$$\int_0^T h(X_s, s) dX_s = H(X_T, T) - H(X_0, 0) - \int_0^T \left[ H_t(X_s, s) + \frac{h_x(X_s, s)}{2} \right] ds,$$

where $H_x(x, t) = h(x, t)$, i.e., $H$ is an antiderivative of $h$ (in $x$). Therefore,

$$e^{\int_0^T h(X_s, s) dX_s} = e^{H(X_T, T) - H(X_0, 0) - \int_0^T \left[ H_t(X_s, s) + \frac{h_x(X_s, s)}{2} \right] ds}.$$
Brownian bridge representation

For $T > 0$, 

$$p(T, y|0, x) = \tilde{E}_y \left[ e^{\int_0^T h(X_s, s) dX_s - \frac{1}{2} \int_0^T h^2(X_s, s) ds} \right] \frac{e^{-\frac{(y-x)^2}{2T}}}{\sqrt{2\pi T}}$$

$$= \frac{e^{-\frac{(y-x)^2}{2T}}}{\sqrt{2\pi T}} e^{H(y, T) - H(x, 0)} \times \tilde{E} \left[ e^{-\frac{1}{2} \int_0^T h^2(X^{x,y}_s, s) + h_x(X^{x,y}_s, s) + 2H_t(X^{x,y}_s, s) ds} \right].$$

$X^{x,y}_s$ is the standard Brownian bridge under $\tilde{P}$ with initial and terminal points at $x$ and $y$ respectively in the time horizon $[0, T]$. 

We consider $h$ in the case that it is independent of time $t$. For small time horizon, we approximate the Brownian bridge expectation as

$$\tilde{\mathbb{E}} \left[ e^{-\frac{1}{2} \int_0^T h^2(X_s^{x,y}) + h'(X_s^{x,y}) \, ds} \right]$$

$$\approx 1 - \frac{1}{2} \int_0^T \tilde{\mathbb{E}} \left[ h^2(X_s^{x,y}) + h'(X_s^{x,y}) \right] \, ds$$

$$\approx 1 - \frac{1}{2} \int_0^T \left[ h^2 \left( x + \frac{s}{T}(y-x) \right) + h' \left( x + \frac{s}{T}(y-x) \right) \right] \, ds$$

$$= 1 - \frac{T}{2(y-x)} \int_x^y (h^2(\xi) + h'(\xi)) \, d\xi$$

- In the second approximation, we estimate Brownian bridge expectation by evaluating along the straight line:
  $$x(s) = x + \frac{s}{T}(y-x).$$

- Note that $\tilde{\mathbb{E}}_y [X_s] = x(s)$ for $0 \leq s \leq T$. 
Recovery of heat kernel expansion

We end up with the following small time approximation of the transition density:

\[
p(T, y|0, x) \sim \frac{e^{-(y-x)^2/2T}}{\sqrt{2\pi T}} e^{H(y)-H(x)} \times \\
\left[ 1 - \frac{T}{2(y-x)} \int_x^y (h^2(\xi) + h'(\xi)) d\xi \right],
\]

which is exactly the heat kernel expansion up to order 1!

- The conventional way in deriving the heat kernel expansion in PDE theory is by applying the WKB ansatz.

\[
p(t, x, y) \sim \frac{1}{t^{n/2}} e^{-\frac{d^2(x,y)}{2t}} \left[ u_0(x, y) + u_1(x, y) t + o(t) \right],
\]

where \( n \) is the dimension of the underlying space.
Fractional Brownian motion (fBM)

A centered Gaussian process $B^H = \{B^H_t; t \in [0, T]\}$ is called a fractional Brownian motion (fBM) with Hurst parameter $H \in (0, 1)$ if it has the covariance function

$$
\mathbb{E}(B^H_s B^H_t) = R_H(s, t) := \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}),
$$

for all $s, t \in [0, T]$.

- Standard Brownian motion corresponds to $H = \frac{1}{2}$.
- $B^H_t$ is not a semimartingale when $H \neq \frac{1}{2}$.
- $B^H_t$ is Hölder continuous of order $\beta$ in $t$ for any $\beta < H$ almost surely.
Representation of fBM

A fBM $B^H$ can be expressed in terms of a stochastic integral with respect to standard Brownian motion as

$$B^H_t = \int_0^t K_H(t, s) dB_s,$$

where $K_H$ is given by

$$K_H(t, s) = c_H \left[ \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H - \frac{1}{2})s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} \right] 1_{[0,t]}.$$

Consider the Molchan-Golosov operator

$$(K_H f)(t) = \int_0^T K_H(t, s)f(s)ds, \quad f \in L^2([0, T]). \quad (1)$$
Consider the two dimensional stochastic system

\[
\begin{align*}
X_t &= x_0 + \rho B_t + \sqrt{1 - \rho^2} W_t + \int_0^t h_1(s, X_s, Y_s) ds, \\
Y_t &= y_0 + B_t^H + \int_0^t h_2(s, X_s, Y_s) ds,
\end{align*}
\]

(2)

where \((X_0, Y_0) = (x_0, y_0)\) is the initial point, \(\rho \in (0, 1)\), and the two functions \(h_1, h_2 : [0, T] \times \mathbb{R}^2 \to \mathbb{R}\) are deterministic.

- \(X_t\) is a standard Brownian motion with drift
- \(Y_t\) is a fractional Brownian motion with drift
- Goal: Obtain a bridge representation for the joint density of \((X_T, Y_T)\) and a small time approximation accordingly
Assumptions

(a) The functions $h_1$ and $h_2$ are Lipschitz in $x, y$ uniformly for $t$. That is, there exists a constant $L > 0$ such that

$$|h_i(t, x_1, y_1) - h_i(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|), \quad i = 1, 2,$$

(3)

for all $t \in [0, T]$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

(b) (i) If $H > \frac{1}{2}$, there exist two constants $L > 0$ and $\gamma \in (H - \frac{1}{2}, \frac{1}{2})$ such that the function $h_1$ satisfies

$$|h_1(t, 0, 0)| \leq L, \quad \forall t \in [0, T],$$

and the function $h_2$ satisfies

$$|h_2(t, x, y) - h_2(s, x, y)| \leq L|t - s|^\gamma, \quad \forall s, t \in [0, T], \forall (x, y) \in \mathbb{R}^2,$$

(4)

i.e., $h_2$ is Hölder continuous in $t$ of order $\gamma$ uniformly for $x$ and $y$.

(ii) If $H \leq \frac{1}{2}$, there exists a constant $L > 0$ such that

$$|h_i(t, 0, 0)| \leq L, \quad \forall s, t \in [0, T], \quad i = 1, 2.$$
Theorem

Let the conditions in the assumptions be satisfied. Then, there exists a positive constant $\delta$ such that the system (2) has a unique solution $(X, Y)$ when $T < \delta$. Moreover, the trajectories of $X$ and $Y$ satisfy $X \in C^{\frac{1}{2} - \epsilon}([0, T])$ and $Y \in C^{H - \epsilon}([0, T])$ almost surely for every $0 < \epsilon < \min\{\frac{1}{2}, H\}$.

Sketch of Proof: Using contraction mapping theorem: Let $(x^i, y^i), i = 1, 2$, be two stochastic processes taking values in $C([0, T])$. Define

\[
\begin{align*}
X^i_t &= x_0 + \rho B_t + \sqrt{1 - \rho^2} W_t + \int_0^t h_1(s, x^i_s, y^i_s) \, ds, \\
Y^i_t &= y_0 + \beta^H_t + \int_0^t h_2(s, x^i_s, y^i_s) \, ds,
\end{align*}
\]

for each $i = 1, 2$. 
Novikov’s condition

The processes $\tilde{h}_1$ and $\tilde{h}_2$ are determined by the following system of equations

$$\sqrt{1 - \rho^2} \tilde{h}_1(t) + \rho \tilde{h}_2(t) = h_1(t, X_t, Y_t),$$

$$\tilde{h}_2(t) = \mathcal{K}_H^{-1} \left( \int_0^t h_2(s, X_s, Y_s) ds \right) (t),$$

where $\mathcal{K}_H^{-1}$ is the inverse of the Molchan-Golosov operator defined in (1).

Lemma

There exists a small $t_0 > 0$ such that the adapted processes $\tilde{h}_1$ and $\tilde{h}_2$ satisfy the Novikov’s condition in $[0, t_0]$. That is,

$$\mathbb{E} \left[ \exp \left\{ \frac{1}{2} \int_0^{t_0} |\tilde{h}_1(t)|^2 dt + \frac{1}{2} \int_0^{t_0} |\tilde{h}_2(t)|^2 dt \right\} \right] < \infty.$$  (5)
Girsanov’s Theorem

We consider all $T \leq t_0$ and define

$$ \frac{d\tilde{P}}{dP} = \exp \left\{ -\int_0^T \tilde{h}_1(t) dW_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt - \int_0^T \tilde{h}_2(t) dB_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt \right\}, $$

Theorem

Under the probability measure $\tilde{P}$, the processes

$\tilde{W} = \left\{ \tilde{W}_t = W_t + \int_0^t \tilde{h}_1(s) ds, \ t \in [0, T] \right\}$ and

$\tilde{B} = \left\{ \tilde{B}_t = B_t + \int_0^t \tilde{h}_2(s) ds, \ t \in [0, T] \right\}$ become two independent Brownian motions, and the process

$\tilde{B}^H = \left\{ \tilde{B}^H_t = B^H_t + \int_0^t K_{H}(t,s) \tilde{h}_2(s) ds, \ t \in [0, T] \right\}$ becomes a fractional Brownian motion.
We can rewrite the processes \( X \) and \( Y \) as

\[
X_t = x_0 + \rho B_t + \sqrt{1 - \rho^2} W_t + \int_0^t h_1(s, X_s, Y_s) ds
\]

\[
= x_0 + \rho \tilde{B}_t + \sqrt{1 - \rho^2} \tilde{W}_t + \int_0^t \left[ h_1(s, X_s, Y_s) - \rho \tilde{h}_2(s) - \sqrt{1 - \rho^2} \tilde{h}_1(s) \right] ds
\]

\[
= x_0 + \rho \tilde{B}_t + \sqrt{1 - \rho^2} \tilde{W}_t
\]

(6)

and

\[
Y_t = y_0 + B_t^H + \int_0^t h_2(s, X_s, Y_s) ds
\]

\[
= y_0 + \int_0^t K_H(t, s) dB_s + \int_0^t h_2(s, X_s, Y_s) ds
\]

\[
= y_0 + \int_0^t K_H(t, s) d\tilde{B}_s - \int_0^t K_H(t, s) \tilde{h}_2(s) ds + \int_0^t h_2(s, X_s, Y_s) ds
\]

\[
= y_0 + \tilde{B}_t^H.
\]

(7)
Bridge representation for the joint density

**Theorem**

The joint density \( p_T(x, y | x_0, y_0) \) of \((X_T, Y_T)\) at time \( T \) has the bridge representation

\[
p_T(x, y | x_0, y_0) = \phi_T(x - x_0, y - y_0) \times \tilde{E}_{x, y} \left[ e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) dB_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt} \right],
\]

where \( \phi_T \) is the bivariate Gaussian density of \((X_T - x_0, Y_T - y_0)\) under \( \tilde{P} \)

\[
\phi_T(\xi, \eta) = \frac{1}{2\pi T^{H+\frac{1}{2}}} \sqrt{1 - \rho^2 \kappa_H^2} \times \exp \left\{ -\frac{1}{2 \left( 1 - \rho^2 \kappa_H^2 \right)} \left[ \left( \frac{\xi}{\sqrt{T}} \right)^2 - 2\rho \kappa_H \left( \frac{\xi}{\sqrt{T}} \right) \left( \frac{\eta}{TH} \right) + \left( \frac{\eta}{TH} \right)^2 \right] \right\}.
\]
Sketch of Proof: For any bounded function $f$ defined on $\mathbb{R}^2$, we have

$$
\int p_T(x, y | x_0, y_0) f(x, y) dx dy
$$

$$
= \mathbb{E}[f(X_T, Y_T)] = \tilde{\mathbb{E}} \left[ f(X_T, Y_T) \frac{dP}{d\tilde{P}} \right]
$$

$$
= \tilde{\mathbb{E}} \left[ f(X_T, Y_T) e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt} \right]
$$

$$
= \int \phi_T(x - x_0, y - y_0) f(x, y)
$$

$$
\times \tilde{\mathbb{E}}_{x, y} \left[ e^{\int_0^T \tilde{h}_1(t) d\tilde{W}_t - \frac{1}{2} \int_0^T \tilde{h}_1^2(t) dt + \int_0^T \tilde{h}_2(t) d\tilde{B}_t - \frac{1}{2} \int_0^T \tilde{h}_2^2(t) dt} \right] dx dy.
$$
Modal-path approximation

Notations:
\[ \langle f \rangle = \int_0^T f(s) ds, \forall f \in L^1([0, T]). \]

and
\[ \bar{\rho} = \sqrt{1 - \rho^2}, \kappa_H = c_H \frac{B \left( \frac{3}{2} - H, H + \frac{1}{2} \right)}{H + \frac{1}{2}}, \rho_H = \rho \kappa_H, \bar{\rho}_H = \sqrt{1 - \rho_H^2}. \]

Approximating the random paths \( X_s \) and \( Y_s \) by their respective modes (thus the term modal-path) \( \hat{E}[X_s^{x,y}] \) and \( \hat{E}[Y_s^{x,y}] \), where \( X^{x,y} \) and \( Y^{x,y} \) is the Gaussian bridge that connect \( (X_0, Y_0) = (x_0, y_0) \) and \( (X_T, Y_T) = (x, y) \). Then we obtain a small time approximation of the joint probability density \( p_T(x, y|x_0, y_0) \) as \( T \to 0 \) as
\[ p_T(x, y|x_0, y_0) = \phi(x - x_0, y - y_0) e^{\omega(T)} \left\{ 1 + o(T^\alpha) \right\}, \]
where

\[
\omega(T) = \frac{1}{\bar{\rho}^2} \left\{ \left( \frac{\bar{\rho}\langle \hat{h}_1 \rangle}{\sqrt{T}} + \frac{\rho\langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\rho_H \langle \bar{h}_2 \rangle}{T^H} \right) \left( \frac{x - x_0}{\sqrt{T}} \right) \right.

- \rho_H \left( \frac{\bar{\rho}\langle \hat{h}_1 \rangle}{\sqrt{T}} + \frac{\rho\langle \hat{h}_2 \rangle}{\sqrt{T}} - \frac{\rho_H \langle \bar{h}_2 \rangle}{T^H} \right) \left( \frac{y - y_0}{T^H} \right) + \bar{\rho}_H \frac{\langle \bar{h}_2 \rangle}{T^H} \left( \frac{y - y_0}{T^H} \right) \left. \right\},
\]

- When \( H = \frac{1}{2} \), the modal-path approximation recovers the heat kernel expansion up to zeroth order!
Explicit expression for modal-path

The modal-path from \((x_0, y_0)\) to \((x, y)\) has the explicitly expression

\[
\begin{align*}
\tilde{E}[X_t] &= x_0 + m_{11}(t)(x - x_0) + m_{12}(t)(y - y_0), \\
\tilde{E}[Y_t] &= y_0 + m_{21}(t)(x - x_0) + m_{22}(t)(y - y_0),
\end{align*}
\]

where

\[
\begin{align*}
m_{11}(t) &= \frac{1}{\rho_H^2} \left( \frac{t}{T} - \frac{\rho \rho_H}{T^{H+\frac{1}{2}}} \int_0^t K_H(T, s)ds \right), \\
m_{12}(t) &= \frac{1}{\rho_H^2} \left( -\rho_H \frac{t}{T^{H+\frac{1}{2}}} + \frac{\rho}{T^{2H}} \int_0^t K_H(T, s)ds \right), \\
m_{21}(t) &= \frac{\rho_H}{\rho_H^2} \left( \frac{t^{H+\frac{1}{2}}}{T} - \frac{R_H(t, T)}{T^{H+\frac{1}{2}}} \right), \\
m_{22}(t) &= \frac{1}{\rho_H^2} \left( -\rho_H^2 \left\{ \frac{t}{T} \right\}^{H+\frac{1}{2}} + \frac{R_H(t, T)}{T^{2H}} \right).
\end{align*}
\]
How does the modal-path look like?

The plots of modal-paths from \((x_0, y_0) = (0, 0)\) to \((x, y) = (1, 1)\) within the time interval \([0, 1]\) with \(\rho = 0, 0.7\) and Hurst exponents \(H = 0.01, 0.25, 0.49, \text{ and } 0.75\).
How does the modal-path look like?

The plots of modal-paths from \((x_0, y_0) = (0, 0)\) to \((x, y) = (1, 1)\) within the time interval \([0, 1]\) with \(\rho = -0.7, -0.9\) and Hurst exponents \(H = 0.01, 0.25, 0.49,\) and \(0.75\).
THANK YOU FOR YOUR ATTENTION.