Suppose that you invest \( x \) dollars at a nominal rate of \( r \) compounded \( m \) times per year. How soon would your investment double?

This period is called doubling time and the rule of 72 allows one to quickly estimate the doubling time.

But let’s do the exact analysis first. Let \( t \) represent time (in years). Our present value/future value compound interest formula says that

\[
2x = x \left(1 + \frac{r}{m}\right)^{mt}.
\]

Clearly, the actual value of \( x \) is irrelevant (unless \( x = 0 \), which is not much of an investment), and we have

\[
2 = \left(1 + \frac{r}{m}\right)^{mt}.
\]

We would like to solve this equation for \( t \). To do this, apply logarithms because \( t \) is hiding in the exponent:

\[
\ln 2 = \ln \left(\left(1 + \frac{r}{m}\right)^{mt}\right) = mt \ln \left(1 + \frac{r}{m}\right)\]

\[
t = \frac{\ln 2}{m \ln \left(1 + \frac{r}{m}\right)}.
\]

Is this \( t \) the doubling time? No, it is not (unless \( mt \) is an integer, which is a very rare occasion). The correct answer is given by the next (after \( t \)) compounding date. Why?

An example should clarify things. Let \$100 be invested, let 6% be the rate, and let the compounding be done monthly. Then, according to our calculation,

\[
t = \frac{\ln 2}{12 \ln \left(1 + \frac{.06}{12}\right)} = \frac{\ln 2}{12 \ln(1.005)} \approx 11.581 \text{ (years)}.
\]

But how many months is .581 of a year?

\[
.581 \times 12 \approx 6.97 \text{ (months)}.
\]

This means that \( t \) falls between two compounding dates. After 11 years and 6 months, the amount

\[
A = 100(1.005)^{11 \cdot 12 + 6} \approx 199.03
\]
is not yet twice the investment. After 11 years and 7 months, the amount
\[ A = 100(1.005)^{11 \cdot 12 + 7} \approx 200.02 \]
goes over $200. But since there was no compounding done in between, we choose:

doubling time = 11 years and 7 months.

Alternatively, we could solve for \( mt \), which is the number of compounding periods,
\[ mt = \frac{\ln 2}{\ln (1 + \frac{r}{m})} = \frac{\log(2)}{\log(1.005)} = 138.9757 \ldots , \]
round up to 139, and then convert 139 periods to (in our case) years and months:
\[ 139 \text{ months} = 11 \text{ years and 7 months}. \]

OK, in either way of solution some appropriate rounding has to be done.

Now what about the rule of 72? This rule says that
doubling time \( \approx \frac{72}{100 \ r} \) (years).

To explain it we need two facts. First,
\[ \ln(2) \approx .6931478 , \]
which is easy to check. Second, if \( r/m \) is small,
\[ m \ln \left(1 + \frac{r}{m}\right) \approx r \]
(this comes from Calculus).

So does this give
\[ t = \frac{\ln 2}{m \ln \left(1 + \frac{r}{m}\right)} \approx \frac{.693}{r} = \frac{69.3}{100 \ r} \ ? \]
Yes.

However, (being a multiple of 2, 3, 4, 6, 12, 24) 72 is more convenient computationally than 69.3. Also, \( m \ln(1 + \frac{r}{m}) \approx r - \frac{r^2}{2m} \) is, in fact, less than \( r \). So one chooses a slightly larger numerator and a slightly larger denominator in hope that the quotient comes out to be a fair approximation:
\[ t = \frac{\ln 2}{m \ln \left(1 + \frac{r}{m}\right)} \approx \frac{.72}{r} = \frac{72}{100 \ r} . \]

In our example, the rule of 72 gives:
doubling time \( \approx \frac{72}{100 \ r} = \frac{72}{6} = 12 \) (years),
which is not that accurate (5 months off!), but quickly gives an idea of how long one should wait.

It is also instructive to look at the case of continuous compounding. We have:

\[
2x = xe^{rt} \\
2 = e^{rt} \\
\ln(2) = rt \\
t = \frac{\ln(2)}{r}.
\]

This answer is exact, \(\ln(2)/r\) is the doubling time. For computational purposes, we may take

\[
\text{doubling time} \approx \frac{.693}{r} = \frac{69.3}{100 \ r}.
\]