3. In each part, obtain the first four nonzero terms of the Maclaurin series for the function by making an appropriate substitution in one of the binomial series obtained in Example 4 of Section 9.9.

(a) $(2+x)^{-1/2}$ (b) $(1-x^2)^{-2}$

4. (a) Use the Maclaurin series for $1/(1-x)$ to find the Maclaurin series for $1/(a-x)$, where $a \neq 0$, and state the radius of convergence of the series.

(b) Use the binomial series for $1/(1+x)^2$ obtained in Example 4 of Section 9.9 to find the first four nonzero terms in the Maclaurin series for $1/(a+x)^2$, where $a \neq 0$, and state the radius of convergence of the series.

5–8 Find the first four nonzero terms of the Maclaurin series for the function by making an appropriate substitution in a known Maclaurin series and performing any algebraic operations that are required. State the radius of convergence of the series. ■

5. (a) $\sin 2x$ (b) e^{-2x} (c) e^{x^2} (d) $x^2 \cos \pi x$

6. (a) $\cos 2x$ (b) $x^2 e^x$ (c) xe^{-x} (d) $\sin(x^2)$

7. (a) $\dfrac{x^2}{1+3x}$ (b) $x \sinh 2x$ (c) $x(1-x^2)^{3/2}$

8. (a) $\dfrac{x}{x-1}$ (b) $3\cosh(x^2)$ (c) $\dfrac{x}{(1+2x)^3}$

9–10 Find the first four nonzero terms of the Maclaurin series for the function by using an appropriate trigonometric identity or property of logarithms and then substituting in a known Maclaurin series. ■

9. (a) $\sin^2 x$ (b) $\ln[(1+x^3)^{12}]$

10. (a) $\cos^2 x$ (b) $\ln\left(\dfrac{1-x}{1+x}\right)$

11. (a) Use a known Maclaurin series to find the Taylor series of $1/x$ about $x=1$ by expressing this function as

$$\frac{1}{x} = \frac{1}{1-(1-x)}$$

(b) Find the interval of convergence of the Taylor series.

12. Use the method of Exercise 11 to find the Taylor series of $1/x$ about $x = x_0$, and state the interval of convergence of the Taylor series.

13–14 Find the first four nonzero terms of the Maclaurin series for the function by multiplying the Maclaurin series of the factors. ■

13. (a) $e^x \sin x$ (b) $\sqrt{1+x}\ln(1+x)$

14. (a) $e^{-x^2}\cos x$ (b) $(1+x^2)^{4/3}(1+x)^{1/3}$

15–16 Find the first four nonzero terms of the Maclaurin series for the function by dividing appropriate Maclaurin series. ■

15. (a) $\sec x \left(= \dfrac{1}{\cos x}\right)$ (b) $\dfrac{\sin x}{e^x}$

16. (a) $\dfrac{\tan^{-1} x}{1+x}$ (b) $\dfrac{\ln(1+x)}{1-x}$

17. Use the Maclaurin series for e^x and e^{-x} to derive the Maclaurin series for $\sinh x$ and $\cosh x$. Include the general terms in your answers and state the radius of convergence of each series.

18. Use the Maclaurin series for $\sinh x$ and $\cosh x$ to obtain the first four nonzero terms in the Maclaurin series for $\tanh x$.

19–20 Find the first five nonzero terms of the Maclaurin series for the function by using partial fractions and a known Maclaurin series. ■

19. $\dfrac{4x-2}{x^2-1}$ **20.** $\dfrac{x^3+x^2+2x-2}{x^2-1}$

21–22 Confirm the derivative formula by differentiating the appropriate Maclaurin series term by term. ■

21. (a) $\dfrac{d}{dx}[\cos x] = -\sin x$ (b) $\dfrac{d}{dx}[\ln(1+x)] = \dfrac{1}{1+x}$

22. (a) $\dfrac{d}{dx}[\sinh x] = \cosh x$ (b) $\dfrac{d}{dx}[\tan^{-1} x] = \dfrac{1}{1+x^2}$

23–24 Confirm the integration formula by integrating the appropriate Maclaurin series term by term. ■

23. (a) $\displaystyle\int e^x\,dx = e^x + C$

(b) $\displaystyle\int \sinh x\,dx = \cosh x + C$

24. (a) $\displaystyle\int \sin x\,dx = -\cos x + C$

(b) $\displaystyle\int \frac{1}{1+x}\,dx = \ln(1+x) + C$

25. Consider the series

$$\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)(k+2)}$$

Determine the intervals of convergence for this series and for the series obtained by differentiating this series term by term.

26. Consider the series

$$\sum_{k=1}^{\infty} \frac{(-3)^k}{k} x^k$$

Determine the intervals of convergence for this series and for the series obtained by integrating this series term by term.

27. (a) Use the Maclaurin series for $1/(1-x)$ to find the Maclaurin series for

$$f(x) = \frac{x}{1-x^2}$$

(b) Use the Maclaurin series obtained in part (a) to find $f^{(5)}(0)$ and $f^{(6)}(0)$.

(c) What can you say about the value of $f^{(n)}(0)$?

28. Let $f(x) = x^2 \cos 2x$. Use the method of Exercise 27 to find $f^{(99)}(0)$.

29–30 The limit of an indeterminate form as $x \to x_0$ can sometimes be found by expanding the functions involved in Taylor series about $x = x_0$ and taking the limit of the series term by term. Use this method to find the limits in these exercises. ■

29. (a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$ (b) $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - x}{x^3}$

30. (a) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{\sin x}$ (b) $\displaystyle\lim_{x\to 0} \frac{\ln\sqrt{1+x} - \sin 2x}{x}$

31–34 Use Maclaurin series to approximate the integral to three decimal-place accuracy. ■

31. $\int_0^1 \sin(x^2)\,dx$

32. $\int_0^{1/2} \tan^{-1}(2x^2)\,dx$

33. $\int_0^{0.2} \sqrt[3]{1+x^4}\,dx$

34. $\int_0^{1/2} \frac{dx}{\sqrt[4]{x^2+1}}$

FOCUS ON CONCEPTS

35. (a) Find the Maclaurin series for e^{x^4}. What is the radius of convergence?

(b) Explain two different ways to use the Maclaurin series for e^{x^4} to find a series for $x^3e^{x^4}$. Confirm that both methods produce the same series.

36. (a) Differentiate the Maclaurin series for $1/(1-x)$, and use the result to show that

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad \text{for } -1 < x < 1$$

(b) Integrate the Maclaurin series for $1/(1-x)$, and use the result to show that

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x) \quad \text{for } -1 < x < 1$$

(c) Use the result in part (b) to show that

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^k}{k} = \ln(1+x) \quad \text{for } -1 < x < 1$$

(d) Show that the series in part (c) converges if $x = 1$.

(e) Use the remark following Example 3 to show that

$$\sum_{k=1}^{\infty} (-1)^{k+1}\frac{x^k}{k} = \ln(1+x) \quad \text{for } -1 < x \le 1$$

37. Use the results in Exercise 36 to find the sum of the series.

(a) $\sum_{k=1}^{\infty} \frac{k}{3^k} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots$

(b) $\sum_{k=1}^{\infty} \frac{1}{k(4^k)} = \frac{1}{4} + \frac{1}{2(4^2)} + \frac{1}{3(4^3)} + \frac{1}{4(4^4)} + \cdots$

38. Use the results in Exercise 36 to find the sum of each series.

(a) $\sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

(b) $\sum_{k=1}^{\infty} \frac{(e-1)^k}{ke^k} = \frac{e-1}{e} + \frac{(e-1)^2}{2(e^2)} - \frac{(e-1)^3}{3(e^3)} + \cdots$

39. (a) Use the relationship

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}x + C$$

to find the first four nonzero terms in the Maclaurin series for $\sinh^{-1}x$.

(b) Express the series in sigma notation.

(c) What is the radius of convergence?

40. (a) Use the relationship

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

to find the first four nonzero terms in the Maclaurin series for $\sin^{-1}x$.

(b) Express the series in sigma notation.

(c) What is the radius of convergence?

41. We showed by Formula (19) of Section 8.2 that if there are y_0 units of radioactive carbon-14 present at time $t = 0$, then the number of units present t years later is

$$y(t) = y_0e^{-0.000121t}$$

(a) Express $y(t)$ as a Maclaurin series.

(b) Use the first two terms in the series to show that the number of units present after 1 year is approximately $(0.999879)y_0$.

(c) Compare this to the value produced by the formula for $y(t)$.

C **42.** Suppose that a simple pendulum with a length of $L = 1$ meter is given an initial displacement of $\theta_0 = 5°$ from the vertical.

(a) Approximate the period T of the pendulum using Formula (9) for the first-order model of T. [*Note:* Take $g = 9.8\ \text{m/s}^2$.]

(b) Approximate the period of the pendulum using Formula (10) for the second-order model.

(c) Use the numerical integration capability of a CAS to approximate the period of the pendulum from Formula (7), and compare it to the values obtained in parts (a) and (b).

43. Use the first three nonzero terms in Formula (8) and the Wallis sine formula in the Endpaper Integral Table (Formula 122) to obtain a model for the period of a simple pendulum.

44. Recall that the gravitational force exerted by the Earth on an object is called the object's *weight* (or more precisely, its *Earth weight*). If an object of mass m is on the surface of the Earth (mean sea level), then the magnitude of its weight is mg, where g is the acceleration due to gravity at the Earth's surface. A more general formula for the magnitude of the gravitational force that the Earth exerts on an object of mass m is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the Earth and h is the height of the object above the Earth's surface.

(a) Use the binomial series for $1/(1+x)^2$ obtained in Example 4 of Section 9.9 to express F as a Maclaurin series in powers of h/R.

(b) Show that if $h = 0$, then $F = mg$.

(c) Show that if $h/R \approx 0$, then $F \approx mg - (2mgh/R)$. [*Note:* The quantity $2mgh/R$ can be thought of as a "correction term" for the weight that takes the object's height above the Earth's surface into account.]

(d) If we assume that the Earth is a sphere of radius $R = 4000$ mi at mean sea level, by approximately what percentage does a person's weight change in going from mean sea level to the top of Mt. Everest (29,028 ft)?

45. (a) Show that the Bessel function $J_0(x)$ given by Formula (4) of Section 9.8 satisfies the differential equation $xy'' + y' + xy = 0$. (This is called the ***Bessel equation of order zero***.)

(cont.)