EXERCISE SET 9.2

1–6 Use the difference $a_{n+1} - a_n$ to show that the given sequence $\{a_n\}$ is strictly increasing or strictly decreasing.

1. $\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$ **2.** $\left\{1 - \frac{1}{n}\right\}_{n=1}^{+\infty}$ **3.** $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$

4. $\left\{\frac{n}{4n-1}\right\}_{n=1}^{+\infty}$ **5.** $\{n - 2^n\}_{n=1}^{+\infty}$ **6.** $\{n - n^2\}_{n=1}^{+\infty}$

7–12 Use the ratio a_{n+1}/a_n to show that the given sequence $\{a_n\}$ is strictly increasing or strictly decreasing.

7. $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ **8.** $\left\{\frac{2^n}{1+2^n}\right\}_{n=1}^{+\infty}$ **9.** $\{ne^{-n}\}_{n=1}^{+\infty}$

10. $\left\{\frac{10^n}{(2n)!}\right\}_{n=1}^{+\infty}$ **11.** $\left\{\frac{n^n}{n!}\right\}_{n=1}^{+\infty}$ **12.** $\left\{\frac{5^n}{2^{(n^2)}}\right\}_{n=1}^{+\infty}$

13–16 True–False Determine whether the statement is true or false. Explain your answer.

13. If $a_{n+1} - a_n > 0$ for all $n \geq 1$, then the sequence $\{a_n\}$ is strictly increasing.

14. A sequence $\{a_n\}$ is monotone if $a_{n+1} - a_n \neq 0$ for all $n \geq 1$.

15. Any bounded sequence converges.

16. If $\{a_n\}$ is eventually increasing, then $a_{100} < a_{200}$.

17–20 Use differentiation to show that the given sequence is strictly increasing or strictly decreasing.

17. $\left\{\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$ **18.** $\left\{\frac{\ln(n+2)}{n+2}\right\}_{n=1}^{+\infty}$

19. $\{\tan^{-1} n\}_{n=1}^{+\infty}$ **20.** $\{ne^{-2n}\}_{n=1}^{+\infty}$

21–24 Show that the given sequence is eventually strictly increasing or eventually strictly decreasing.

21. $\{2n^2 - 7n\}_{n=1}^{+\infty}$ **22.** $\left\{\frac{n}{n^2+10}\right\}_{n=1}^{+\infty}$

23. $\left\{\frac{n!}{3^n}\right\}_{n=1}^{+\infty}$ **24.** $\{n^5 e^{-n}\}_{n=1}^{+\infty}$

FOCUS ON CONCEPTS

25. Suppose that $\{a_n\}$ is a monotone sequence such that $1 \leq a_n \leq 2$ for all n. Must the sequence converge? If so, what can you say about the limit?

26. Suppose that $\{a_n\}$ is a monotone sequence such that $a_n \leq 2$ for all n. Must the sequence converge? If so, what can you say about the limit?

27. Let $\{a_n\}$ be the sequence defined recursively by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.
(a) List the first three terms of the sequence.
(b) Show that $a_n < 2$ for $n \geq 1$.
(c) Show that $a_{n+1}^2 - a_n^2 = (2 - a_n)(1 + a_n)$ for $n \geq 1$.
(d) Use the results in parts (b) and (c) to show that $\{a_n\}$ is a strictly increasing sequence. [*Hint:* If x and y are positive real numbers such that $x^2 - y^2 > 0$, then it follows by factoring that $x - y > 0$.]
(e) Show that $\{a_n\}$ converges and find its limit L.

28. Let $\{a_n\}$ be the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \frac{1}{2}[a_n + (3/a_n)]$ for $n \geq 1$.
(a) Show that $a_n \geq \sqrt{3}$ for $n \geq 2$. [*Hint:* What is the minimum value of $\frac{1}{2}[x + (3/x)]$ for $x > 0$?]
(b) Show that $\{a_n\}$ is eventually decreasing. [*Hint:* Examine $a_{n+1} - a_n$ or a_{n+1}/a_n and use the result in part (a).]
(c) Show that $\{a_n\}$ converges and find its limit L.

29–30 The Beverton–Holt model is used to describe changes in a population from one generation to the next under certain assumptions. If the population in generation n is given by x_n, the Beverton–Holt model predicts that the population in the next generation satisfies

$$x_{n+1} = \frac{RKx_n}{K + (R-1)x_n}$$

for some positive constants R and K with $R > 1$. These exercises explore some properties of this population model.

29. Let $\{x_n\}$ be the sequence of population values defined recursively by $x_1 = 60$, and for $n \geq 1, x_{n+1}$ is given by the Beverton–Holt model with $R = 10$ and $K = 300$.
(a) List the first four terms of the sequence $\{x_n\}$.
(b) If $0 < x_n < 300$, show that $0 < x_{n+1} < 300$. Conclude that $0 < x_n < 300$ for $n \geq 1$.
(c) Show that $\{x_n\}$ is increasing.
(d) Show that $\{x_n\}$ converges and find its limit L.

30. Let $\{x_n\}$ be a sequence of population values defined recursively by the Beverton–Holt model for which $x_1 > K$. Assume that the constants R and K satisfy $R > 1$ and $K > 0$.
(a) If $x_n > K$, show that $x_{n+1} > K$. Conclude that $x_n > K$ for all $n \geq 1$.
(b) Show that $\{x_n\}$ is decreasing.
(c) Show that $\{x_n\}$ converges and find its limit L.

31. The goal of this exercise is to establish Formula (5), namely,

$$\lim_{n\to+\infty} \frac{x^n}{n!} = 0$$

Let $a_n = |x|^n/n!$ and observe that the case where $x = 0$ is obvious, so we will focus on the case where $x \neq 0$.
(a) Show that

$$a_{n+1} = \frac{|x|}{n+1} a_n$$

(b) Show that the sequence $\{a_n\}$ is eventually strictly decreasing.
(c) Show that the sequence $\{a_n\}$ converges.

32. (a) Compare appropriate areas in the accompanying figure to deduce the following inequalities for $n \geq 2$:

$$\int_1^n \ln x\,dx < \ln n! < \int_1^{n+1} \ln x\,dx$$

(cont.)