Alternating Series and Leibniz's Test

Let $a_1, a_2, a_3, \dots$ be a sequence of positive numbers. A series of the form

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

is said to be alternating because of the alternating sign pattern. (The series $-a_1 + a_2 - a_3 + \dots$ is also alternating, but it is more reassuring to start summation with a positive term.)

The partial sums S_n of an alternating series are evidently not monotone,

$$S_1 > S_2, \quad S_2 < S_3, \quad S_3 > S_4, \quad \dots .$$

However, the subsequences of odd-numbered and of even-numbered partial sums

$$S_1, S_3, S_5, \dots, \qquad S_2, S_4, S_6, \dots,$$

may exhibit monotonic behavior. In fact, S_{2n+1} and S_{2n} are monotone if and only if the original sequence $a_1, a_2, a_3, \dots$ is monotone.

If convergent, an alternating series may not be absolutely convergent. For this case one has a special test to detect convergence.

ALTERNATING SERIES TEST (Leibniz). *If $a_1, a_2, a_3, \dots$ is a sequence of positive numbers monotonically decreasing to 0, then the series*

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \dots$$

converges.

It is not difficult to prove Leibniz's test. Indeed, since

$$a_1 \geq a_2 \geq a_3 \geq \dots,$$

we have

$$a_1 \geq a_1 - a_2 + a_3 \geq a_1 - a_2 + a_3 - a_4 + a_5 \geq \dots$$
$$a_1 - a_2 \leq a_1 - a_2 + a_3 - a_4 \leq a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \leq \dots ,$$

which means that S_{2n+1} is monotone decreasing and S_{2n} is monotone increasing. Also $S_{2n+1} = S_{2n} + a_{2n+1} > S_{2n}$ for every n, implying that both sequences are bounded and hence convergent. To see that S_{2n+1} and S_{2n} converge to the same limit, observe that $\lim_{n\to\infty}(S_{2n+1} - S_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$. Proof finished.

A couple of conclusions follow from the above argument. First,

$$S_{2n} < S < S_{2n+1},$$

where S is the sum of the series. And second,

$$S - S_{2n} < a_{2n+1}, \quad S_{2n-1} - S < a_{2n}.$$

EXAMPLE. The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges by Leibniz's test. Indeed, the sign pattern is $+ - + - + \ldots$ and, as $n \to \infty$, the term $\frac{1}{n}$ monotonically decreases to 0.

To illustrate the error estimate, observe for instance that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \approx .746$$

is larger than the true sum but by no more than 0.1.