Show all your work on the exam paper, legibly and in detail, to receive full credit. The use of a calculator or any other electronic device is prohibited. You may only use techniques discussed to date in class. You must simplify all answers unless you are explicitly instructed not to.

1. (10 points) Find the parametric equations of the line that passes through the point $(6, 2, 0)$ and is perpendicular to the plane $4y + 3z = -5$.

\[
\vec{n} = \langle 0, 4, 3 \rangle
\]

A normal to the plane is \( \vec{n} = \langle 0, 4, 3 \rangle \) which is parallel to the requested line.

\[
\text{Line: } \begin{cases} 
    x = 6 + 0t = 6 \\
    y = 2 + 4t = 2 + 4t \\
    z = 0 + 3t = 3t
\end{cases}
\]
2. (10 points) Find a unit vector in the direction in which \( f(x, y) = \sin(3x - y) \)

increases most rapidly at \( \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \), and find the rate of change of \( f \) at \( \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \) in that direction.

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 3 \cos(3x - y) \\
\frac{\partial f}{\partial x} \left( \frac{\pi}{4}, \frac{\pi}{2} \right) &= 3 \cos \left( \frac{\pi}{2} \right) = \frac{3\sqrt{2}}{2} \\
\frac{\partial f}{\partial y} &= -\cos(3x - y) \\
\frac{\partial f}{\partial y} \left( \frac{\pi}{4}, \frac{\pi}{2} \right) &= -\cos \left( \frac{\pi}{2} \right) = -\frac{\sqrt{2}}{2}
\end{align*}
\]

\[\nabla f \left( \frac{\pi}{4}, \frac{\pi}{2} \right) = \left< \frac{3\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right>\]

Unit vector in direction in which \( f \) increases most rapidly at \( \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \):

\[
\frac{\left< \frac{3\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right>}{\sqrt{\frac{18}{4} + \frac{2}{4}}} = \frac{\left< \frac{3\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right>}{\sqrt{5}} = \left< \frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right>
\]

Rate of change in this direction is

\[\| \nabla f \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \| = \sqrt{5} \]
3. (10 points) Compute \( \frac{\partial z}{\partial x} \) using implicit differentiation. Leave your answer in terms of \( x, y \) and \( z \).

\[ yz^2 + e^{xz} = y \]

\[ \frac{\partial}{\partial x} \left[ yz^2 + e^{xz} \right] = \frac{\partial}{\partial x} [y] \]

\[ y \left( \frac{\partial z}{\partial x} \right) + e^{xz} \left[ x \frac{\partial z}{\partial x} + z \right] = 0 \]

\[ 2yz \frac{\partial z}{\partial x} + xe^{xz} \frac{\partial z}{\partial x} + ze^{xz} = 0 \]

\[ \frac{\partial z}{\partial x} \left[ 2yz + xe^{xz} \right] = -ze^{xz} \]

\[ \frac{\partial z}{\partial x} = \frac{-ze^{xz}}{2yz + xe^{xz}} \]
4. (10 points) (1 pt each) Match the equations on this page with the picture letters on the opposite page. Not all equations have a match; if no match, write NM. Some pictures may not have a match.

Version #1 (Note: These are the same equations as version #2 but in a different order.)

1 Eqn: \(-y^2 - x^2 + z^2 = 0\) 

2 Eqn: \(x^2 + y^2 + \frac{z^2}{4} = 1\)

3 Eqn: \(-y^2 - x^2 + z^2 = 1\)

4 Eqn: \(x^2 + \frac{y^2}{4} + z^2 = 1\)

5 Eqn: \(y - x^2 - z^2 = 0\)

6 Eqn: \(x^2 + y^2 + z^2 = 4\)

7 Eqn: \(y^2 - x^2 + z^2 = -1\)

8 Eqn: \(-x^2 + z^2 + y^2 = 1\)

9 Eqn: \(x^2 - y^2 + z^2 = 1\)

10 Eqn: \(-y^2 + x^2 + z^2 = 0\)
4. (10 points)(1 pt each) Match the equations on this page with the picture letters on the opposite page. Not all equations have a match; if no match, write NM. Some pictures may not have a match.

Version #2 (Note: These are the same equations as version #1 but in a different order.)

1 Eqn: \(-y^2 - x^2 + z^2 = 0\)  
2 Eqn: \(y^2 - x^2 + z^2 = -1\)  
3 Eqn: \(x^2 + \frac{y^2}{4} + z^2 = 1\)  
4 Eqn: \(-x^2 + z^2 + y^2 = 1\)  
5 Eqn: \(-y^2 + x^2 + z^2 = 0\)  
6 Eqn: \(-y^2 - x^2 + z^2 = 1\)  
7 Eqn: \(x^2 + y^2 + \frac{z^2}{4} = 1\)  
8 Eqn: \(x^2 - y^2 + z^2 = 1\)  
9 Eqn: \(y - x^2 - z^2 = 0\)  
10 Eqn: \(x^2 + y^2 + z^2 = 4\)
5. Consider the parallelepiped with adjacent edges:

\[ \vec{u} = \langle 1, -1, -1 \rangle, \vec{v} = \langle 0, 2, 1 \rangle, \vec{w} = \langle 1, 0, 2 \rangle \]

a. (5 points) Find the area of the face determined by \( \vec{v} \) and \( \vec{w} \).

b. (5 points) Find the angle between \( \vec{u} \) and the plane containing the face determined by \( \vec{v} \) and \( \vec{w} \). You may leave your answer in terms of an inverse trigonometric function.

\[
(a) \quad \vec{v} \times \vec{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 4 \mathbf{j} + 2 \mathbf{k} \\
A = \| \vec{v} \times \vec{w} \| = \sqrt{4^2 + 1^2 + (-2)^2} = \sqrt{21}
\]

(b) To find \( \theta \), we first find \( \alpha \).

\[
\cos \alpha = \frac{\vec{u} \cdot (\vec{v} \times \vec{w})}{\| \vec{u} \| \| \vec{v} \times \vec{w} \|}
\]

\[
\cos \alpha = \frac{\langle 1, -1, -1 \rangle \cdot \langle 4, 1, -2 \rangle}{\sqrt{3} \sqrt{21}}
\]

\[
\cos \alpha = \frac{5}{\sqrt{63}} = \frac{5}{3\sqrt{7}} \quad \alpha = \arccos \left( \frac{5}{3\sqrt{7}} \right)
\]

\[
\theta = \frac{\pi}{2} - \arccos \left( \frac{5}{3\sqrt{7}} \right)
\]
6. (10 points) Evaluate the integral by first reversing the order of integration.

\[ \int_{0}^{4} \int_{\sqrt{y}}^{2} \sqrt{x^3 + 1} \, dx \, dy \]

\[ = \int_{0}^{2} \int_{0}^{x^2} \sqrt{x^3 + 1} \, dy \, dx \]

\[ u = x^3 + 1 \quad x = 2 \Rightarrow u = 9 \]
\[ du = 3x^2 \, dx \quad x = 0 \Rightarrow u = 1 \]
\[ = \frac{1}{3} \left[ \int_{1}^{9} \sqrt{u} \, du \right] = \frac{1}{3} \cdot \frac{2}{3} u^{\frac{3}{2}} \bigg|_{1}^{9} \]
\[ = \frac{2}{9} \left( 9^{\frac{3}{2}} - 1^{\frac{3}{2}} \right) = \frac{2}{9} \left( 27 - 1 \right) = \frac{52}{9} \]
7. (10 points) Find the equation of the tangent plane to the parametric surface 
\( x = uv, \ y = u - v, \ z = u^2 \) at the point where \( u = 1 \) and \( v = 2 \).

\[
\frac{\partial x}{\partial u} = v \quad \frac{\partial y}{\partial u} = 1 \quad \frac{\partial z}{\partial u} = 2u
\]
\[
\vec{r}_u = \langle v, 1, 2u \rangle \quad \vec{r}_u(1, 2) = \langle 2, 1, 2 \rangle
\]
\[
\frac{\partial x}{\partial v} = u \quad \frac{\partial y}{\partial v} = -1 \quad \frac{\partial z}{\partial v} = 0
\]
\[
\vec{r}_v = \langle u, -1, 0 \rangle \quad \vec{r}_v(1, 2) = \langle 1, -1, 0 \rangle
\]

\[
\langle 2, 1, 2 \rangle \times \langle 1, -1, 0 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = \langle 2, 2, -3 \rangle
\]

is normal to the tangent plane.

Point on plane: \( x = uv = (1)(2) = 2 \)
\( y = u - v = 1 - 2 = -1 \)
\( z = u^2 = 1 \)

Tangent plane: \( 2(x-2) + 2(y+1) - 3(z-1) = 0 \)
\[
2x - 4 + 2y + 2 - 3z + 3 = 0
\]
\[
2x + 2y - 3z = -1
\]
8. (10 points) Convert the given iterated integral in rectangular coordinates to an equivalent iterated integral in polar coordinates. Do NOT evaluate the integral.

\[
\int_{0}^{2} \int_{0}^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} \, dy \, dx
\]

\[y = \sqrt{2x-x^2} \implies y^2 = 2x - x^2\]

\[x^2 - 2x + y^2 = 0 \implies (x-1)^2 + y^2 = 1\]

\[\text{Circle Center (1,0) Radius 1}\]

\[r^2 - 2r \cos \theta = 0\]

\[r = 2 \cos \theta\]

\[y = \sqrt{2x-x^2} \quad (r = 2 \cos \theta)\]

\[\int_{0}^{\pi} \int_{0}^{2 \cos \theta} r^2 \, dr \, d\theta = \int_{0}^{\pi} \int_{0}^{2 \cos \theta} \sqrt{r^2} \, dr \, d\theta\]
9. Consider the solid that is bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$

   a. (4 points) Give an iterated integral in rectangular coordinates that represents the volume of the solid. DO NOT EVALUATE THE INTEGRAL.

   $z = \sqrt{16 - x^2 - y^2} \iff z = \sqrt{16 - r^2} \iff \rho = h$

   $z = \sqrt{x^2 + y^2} \iff z = r \iff \phi = \frac{\pi}{4}$

   Intersection: $\sqrt{16 - x^2 - y^2} = \sqrt{x^2 + y^2}$

   $16 = 2x^2 + 2y^2$

   $x^2 + y^2 = 8$

   Projection on $xy$-plane:

   (a) $\int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{\sqrt{8 - x^2}}^{\sqrt{16 - x^2 - y^2}} \int_{-\sqrt{8 - x^2}}^{\sqrt{16 - x^2 - y^2}} 1 \, dz \, dy \, dx$

   (b) $\int_{0}^{2\pi} \int_{0}^{2\sqrt{2}} \int_{r}^{\sqrt{16 - r^2}} r \, dz \, dr \, d\theta$

   (c) $\int_{0}^{\frac{\pi}{4}} \int_{0}^{16} \int_{0}^{\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$
10. (10 points) Use the transformation $u = x + y$, $v = x - y$ to find

$$\iint_R (x - y) e^{x^2 - y^2} \, dA$$

over the rectangular region $R$ enclosed by the lines

$x + y = 0$, $x + y = 1$, $x - y = 1$, $x - y = 4$.

Note: The transformation can be written as $x = \frac{1}{2}(u + v)$, $y = \frac{1}{2}(u - v)$.

$$\frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\iint_S (x - y) e^{x^2 - y^2} \, dA = \iint_S v e^{uv} \left| -\frac{1}{2} \right| \, dA$$

$$= \frac{1}{2} \int_0^1 \int_0^u v e^{uv} \, du \, dv = \frac{1}{2} \int_0^u v \left. e^{uv} \right|_0^1 \, dv$$

$$= \frac{1}{2} \int_0^u (e^v - 1) \, dv = \frac{1}{2} \left[ e^v \right]_0^u - 

= \frac{1}{2} \left[ (e^u - e) - (e - 1) \right] = \frac{1}{2} (e^u - e - 3)$$
11. (bonus) (5 points) Consider a rectangular box that is open at the top and has a volume of 32 ft$^3$. What should the dimensions of the box be for it to have the minimum surface area?

![Diagram of a rectangular box]

Given $xyz = 32$  
$\Rightarrow z = \frac{32}{xy}$  
$x > 0, y > 0, z > 0$

Minimize $S = xy + 2xz + 2yz = xy + \frac{64}{y} + \frac{64}{x}$

$S_x = y - \frac{64}{x^2} = 0 \Rightarrow y = \frac{64}{x^2}$

$S_y = x - \frac{64}{y^2} = 0 \Rightarrow x = \frac{64}{y^2}$

$\Rightarrow x \left(1 - \frac{x^3}{64}\right) = 0 \Rightarrow x = 0, x = 4$ (critical point)

$S_{xx} = 192 \frac{1}{x^3}$  
$S_{xx}(4,4) = 3$  
$S_{xy} = 0$

$S_{yy} = 192 \frac{1}{x^3}$  
$S_{yy}(4,4) = 3$

$\Delta(4,4) = (3)(3) - 0^2 > 0$ and $S_{xx}(4,4) > 0$  
so $(4,4)$ yields a minimum surface area.

Dimensions: $x = 4$ ft, $y = 4$ ft, $z = 2$ ft