Numerical Analysis, Spring 09
Grinshpan

QUIZ 2

A) No, $2 \times 10^{-16}$ has 0 significant digits of $10^{-16}$
B) $\tilde{x} = 1$ has 0 significant digits of $x = 2$, but $\frac{1}{2} = 0.5$ has 1 significant digit of 1.
C) $2.25 \leq x^2 \leq 6.25$ since $1.5 \leq x \leq 2.5$.
D) $0.5 < \sin x \leq 1$ since $1.5 < \frac{x}{2} < 2.5 < \frac{3\pi}{6}$.

EXERCISES 2

1 a) $|\cos(x) - \cos(1.473)| \leq \sin(1.4735) \times 0.0005 < 4.98 \times 10^{-4}$
\[ \left| \frac{\cos(x) - \cos(1.473)}{\cos(x)} \right| \leq \tan(1.4735) \times 0.0005 < 5.2 \times 10^{-3} \]

1 b) $|\arctan(x) - \arctan(2.62)| \leq \frac{1}{1 + 2.615^2} \times 0.005 < 6.38 \times 10^{-4}$
\[ \left| \frac{\arctan(x) - \arctan(2.62)}{\arctan(x)} \right| < \frac{6.38 \times 10^{-4}}{\arctan(2.615)} < 5.3 \times 10^{-4} \]

1 c) $|\ln(x) - \ln(1.4712)| \leq \frac{1}{1.4715} \times 0.0005 < 3.4 \times 10^{-5}$
\[ \left| \frac{\ln(x) - \ln(1.4712)}{\ln(x)} \right| < \frac{3.4 \times 10^{-5}}{\ln(1.4715)} < 8.81 \times 10^{-5} \]

2. By the mean value theorem, $\sin(\sqrt{2}) - \sin(1.414) = \cos(c)(\sqrt{2} - 1.414)$, where $\sqrt{2} < c < 1.414$. Obviously, the error is less than $1 \cdot (\sqrt{2} - 1.414) \approx 0.00022$.
A more accurate error bound would be $\cos(1.414)(\sqrt{2} - 1.414) < 3.335 \times 10^{-5}$.
For comparison, MATLAB gives $\sin(\sqrt{2}) - \sin(1.414) \approx 3.332623 \times 10^{-5}$.
To bound the relative error, write
\[ \frac{\sin(\sqrt{2}) - \sin(1.414)}{\sin(\sqrt{2})} \leq \frac{\cos(1.414)}{\sin(\sqrt{2})} (\sqrt{2} - 1.414) < 3.377 \times 10^{-5}. \]

3. \[ \frac{(1 - \tilde{x}) - (1 - \tilde{y})}{1 - (1 - \tilde{x})} = \frac{\tilde{x} - \tilde{y}}{\tilde{y}} = 1 - \frac{\tilde{y}}{\tilde{x}}. \]
Also $\frac{1}{1 + z + \ldots} \approx 1$ if $z = \text{rel}(\tilde{y})$ has small absolute value.

4. The roots are $20 \pm \sqrt{399}$. Approximate $\sqrt{399}$ by $\xi = 19.975$.
Then $\sqrt{399} - \xi \approx -1.6 \times 10^{-5}$, so $\xi$ has 6 significant digits.
The relative error of $20 + \xi$ is $\frac{\sqrt{399} - \xi}{20 + \sqrt{399}} \approx -3.9 \times 10^{-7}$,
but the relative error of $20 - \xi$ is only $\frac{\xi - \sqrt{399}}{20 - \sqrt{399}} \approx 6.3 \times 10^{-4}$.
To avoid loss of accuracy, use $(20 + \xi)^{-1}$ to estimate the smaller root $(20 + \sqrt{399})^{-1}$. The relative error of $(20 + \xi)^{-1}$ is under $3.9 \times 10^{-7}$.

5. $\sqrt{1 + x} - 1 = \left( \frac{x}{1 + \sqrt{1 + x}} \right) \frac{x}{\sqrt{1 + x}} \ln(1 + x) - \ln x = \ln(1 + \frac{1}{x}), \quad \frac{e^{-1}}{x} \approx 1 + \frac{1}{2} x$. 
A) Form a comparison table (numbers rounded):

<table>
<thead>
<tr>
<th></th>
<th>5/3</th>
<th>8/5</th>
<th>13/8</th>
<th>1.61 = 161/100</th>
<th>1.618 = 860/500</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa =</td>
<td>q\alpha - p</td>
<td>$</td>
<td>.1459</td>
<td>.0002</td>
<td>.0557</td>
</tr>
<tr>
<td>$</td>
<td>\alpha - \frac{p}{q}</td>
<td>$</td>
<td>.0486</td>
<td>.0180</td>
<td>.00697</td>
</tr>
</tbody>
</table>

Evidently, 1.618 is the best of 5, followed by 13/8. For $q \leq 100$, two ratios of Fibonacci numbers, 89/55 and 144/89, yield $|q\alpha - p| < .01$. In fact, $|55\alpha - 89| \approx .0081$, $|\alpha - \frac{89}{55}| \approx .00015$, and $|89\alpha - 144| \approx .005$, $|\alpha - \frac{144}{89}| \approx .000056$.

B) As $x \to 0^+$, $\sqrt{1+x} - 1$ increases to 0.5. The following table was obtained using MATLAB (values rounded). It exhibits noise in evaluation, loss of significant digits, and underflow. The last column gives an accurate alternative.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sqrt{1+x} - 1$</th>
<th>error</th>
<th>signif. digits</th>
<th>$\frac{1}{1+\sqrt{1+x}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-5}$</td>
<td>0.499998750</td>
<td>0.000001250</td>
<td>5</td>
<td>0.499998750000625</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>0.499998750</td>
<td>0.000001250</td>
<td>6</td>
<td>0.499998750000625</td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>0.499999886</td>
<td>0.000000126</td>
<td>7</td>
<td>0.499999987500000</td>
</tr>
<tr>
<td>$10^{-8}$</td>
<td>0.500000415</td>
<td>-0.000000041</td>
<td>7</td>
<td>0.499999998750000</td>
</tr>
<tr>
<td>$10^{-9}$</td>
<td>0.500000041</td>
<td>-0.000000041</td>
<td>7</td>
<td>0.499999999875000</td>
</tr>
<tr>
<td>$10^{-10}$</td>
<td>0.500000041</td>
<td>-0.000000041</td>
<td>7</td>
<td>0.499999999987500</td>
</tr>
<tr>
<td>$10^{-11}$</td>
<td>0.500000041</td>
<td>-0.000000041</td>
<td>7</td>
<td>0.499999999999875</td>
</tr>
<tr>
<td>$10^{-12}$</td>
<td>0.500044500</td>
<td>-0.000000004</td>
<td>4</td>
<td>0.499999999999998</td>
</tr>
<tr>
<td>$10^{-13}$</td>
<td>0.499603611</td>
<td>0.000000003</td>
<td>3</td>
<td>0.499999999999999</td>
</tr>
<tr>
<td>$10^{-14}$</td>
<td>0.488498131</td>
<td>0.011501869</td>
<td>1</td>
<td>0.500000000000000</td>
</tr>
<tr>
<td>$10^{-15}$</td>
<td>0.444089210</td>
<td>0.055910790</td>
<td>0</td>
<td>0.500000000000000</td>
</tr>
<tr>
<td>$10^{-16}$</td>
<td>0</td>
<td>0.5</td>
<td>0</td>
<td>0.500000000000000</td>
</tr>
</tbody>
</table>

C) a) $1.10615 \leq x < 1.10625$, $0.9465 \leq y < 0.9475$, $2.05265 \leq x + y < 2.05375$

b) $23.455 \leq x < 23.465$, $12.7525 \leq y < 12.7535$, $10.7015 < x - y < 10.7125$

c) $2.7465 \leq x < 2.7475$, $6.825 \leq y < 6.835$, $18.7448625 \leq xy < 18.7791625$

d) $8.4725 \leq x < 8.4735$, $0.0635 \leq y < 0.0645$, $\frac{8.4725}{0.0635} < x/y < \frac{8.4735}{0.0645}$