Quiz 6.

A) \( \sqrt{x + 1} - x = x \) is equivalent to \( x^2 - x - 1 = 0, x > 0 \). Hence \( \alpha = \frac{1 + \sqrt{5}}{2} \).

B) \( F'(x) = \frac{1}{2 \sqrt{x + 1}} > 0 \) has a maximum of \( \frac{1}{2 \sqrt{2}} \) on \([1, 2]\).

C) \( F(x) \) maps \([1, 2]\) continuously onto \([\sqrt{2}, \sqrt{3}]\), a proper subset of \([1, 2]\).

D) \( F(x) = \alpha + F'(\alpha)(x - \alpha) + \) higher order terms, for \( x \) close to \( \alpha \).

Hence \( x_{n+1} - \alpha \sim F'(\alpha)(x_n - \alpha) \), the order is 1, the rate is \( F'(\alpha) = \frac{1}{1 + \sqrt{5}} \).

Exercises, week 10.

1. The positive root \( \alpha = \sqrt{5} \) of the equation \( x^2 - 5 = 0 \) lies between 2 and 3. Let’s rewrite our equation in the form \( x = F(x) \) to run the fixed point iterations. We can do this in many different ways. For instance, \( x = 5/x \) with \( F(x) = 5/x \), \( x = \frac{1}{2}(x + \frac{3}{2}) \) with \( F(x) = \frac{1}{2}(x + \frac{3}{2}) \), or \( x = x^2 + x - 5 \) with \( F(x) = x^2 + x - 5 \).

Each of the preceding functions \( F(x) \) has \( \alpha \) as a fixed point, and each of them is continuous and differentiable on the interval \([2, 3]\). However, if we would like to use the interval \([2, 3]\) in our analysis, then \( F(x) = 5/x \) and \( F(x) = x^2 + x - 5 \) do not suit the purpose as they fail to map \([2, 3]\) into \([2, 3]\) (do you see why?).

The function \( F(x) = \frac{1}{2}(x + \frac{3}{2}) \) does map \([2, 3]\) into \([2, 3]\). In fact, \( F(x) \) has a local minimum at \( x = \alpha \) and it maps \([2, 3]\) onto \([\alpha, 2\frac{1}{2}]\), a proper subset of \([2, 3]\). To apply the Contraction Mapping Theorem and conclude convergence estimates, it remains to check that \( |F'(x)| \) is bounded by some \( \lambda < 1 \). This is indeed the case: \( F'(x) = \frac{1}{2}(1 - \frac{1}{x}) \) and so \( \max_{2 \leq x \leq 3} |F'(x)| = \frac{2}{3} < 1 \).

All conditions of the Contraction Mapping theorem are satisfied for \( F(x) = \frac{1}{2}(x + \frac{3}{2}) \) on \([2, 3]\). Therefore, \( \alpha \) is the unique fixed point of \( F(x) = \frac{1}{2}(x + \frac{3}{2}) \) on \([2, 3]\). For any \( x_0 \) in \([2, 3]\) the fixed point iterations \( x_{n+1} = \frac{1}{2}(x_n + 5/x_n) \) converge to \( \alpha \) and \( |x_n - \alpha| \leq \left(\frac{2}{3}\right)^n |x_0 - \alpha| \leq \left(\frac{2}{3}\right)^n \).

The obtained bound \( |x_n - \alpha| < \left(\frac{2}{3}\right)^n \) is simple and convenient. It allows us to easily estimate the number of steps needed to place \( x_n \) within a given distance from \( \alpha \). However, the actual order of convergence is higher. By Taylor expanding \( F(x) = \frac{1}{2}(x + \frac{3}{2}) \) about \( \alpha = \sqrt{5} \), we see that

\[
F(x) = \alpha + \frac{1}{2\sqrt{5}} (x - \alpha)^2 + \text{ higher order terms,}
\]

i.e., the order of convergence is 2 and the rate is \( \frac{1}{2\sqrt{5}} \). This, of course, agrees with our analysis in Homework 5 since the fixed-point iteration for \( F(x) = \frac{1}{2}(x + \frac{3}{2}) \) is identical to the Newton iteration for \( f(x) = x^2 - 5 \).

2. The graph of \( F(x) \) is a parabola symmetric with respect to \( x = \frac{1}{2} \). At the endpoints, \( F(0) = F(1) = 0 \). \( F(x) \) is increasing on \([0, \frac{1}{2}] \), has a maximum of \( \frac{3}{4} \) at \( x = \frac{1}{4} \), and is decreasing on \([\frac{1}{4}, 1] \). Hence \( F(x) \) maps \([0, 1]\) onto \([0, \frac{1}{2}] \subset [0, 1] \) (the
Homework 6.

A) The graph of $F(x)$ is a parabola symmetric with respect to $x = \frac{1}{2}$. At the endpoints, $F(0) = F(1) = 0$. $F(x)$ is increasing on $[0, \frac{1}{2}]$, has a maximum of $\frac{1}{4}$ at $x = \frac{1}{2}$, and is decreasing on $[\frac{1}{2}, 1]$. Hence $F(x)$ maps $[0, 1]$ onto $[0, \frac{1}{4}] \subset [0, 1]$. To find the fixed points, write $x = x - x^2$. This gives $x^2 = 0$, so $x = 0$ and $x = \frac{1}{2}$ are the fixed points.

B) We have $x_{n+1} = x_n(1 - x_n)$, $n \geq 0$. For each $0 \leq t \leq \frac{1}{2}$, the initial values $x_0 = t$ and $x_0 = 1 - t$ result in the same sequence $x_n = x_n(t)$, $n \geq 1$. If $t = 0$ we have the zero sequence. For $t \neq 0$, the sequence is oscillatory and can be shown to slowly converge to $\frac{1}{2}$.

The Contraction Mapping Theorem assumes that $\lambda = \frac{1}{2}$. Here this is not the case and some conclusions of the theorem do not hold: the fixed point is not unique and no estimate of the form $|x_n - \frac{1}{3}| \leq C\lambda^n \to 0$ is possible. However, the convergence still takes place for any $x_0$.

The order of convergence is $p = 1$, the rate is $C = 1$. Indeed, the Taylor expansion of $F(x)$ about $\frac{1}{3}$ is

$$F(x) = \frac{2}{3} F(\frac{2}{3})(x - \frac{2}{3}) + \text{higher order terms},$$

which gives $x_{n+1} = \frac{2}{3} - (x_n - \frac{2}{3})$, $n \to \infty$.

The graphs are shown separately. b) For $-1 \leq x \leq 1$, it is easy to see that $\max \{T_2(x)\} = \max |x| = 1$, $\max \{T_3(x)\} = \frac{x^2}{2} - \frac{1}{2} = \frac{1}{4}$, $\max \{T_4(x)\} = \frac{x^3}{3} - \frac{3x}{4} = \frac{1}{4}$ (examine local/absolute extrema if needed). c) Any horizontal or vertical translation of the graph results in a larger deviation from the $x$-axis.
C) The absolute value of $F'(x) = 1 - 2x$ is maximized at $x = 0$ and $x = 1$, $|F'(0)| = |F'(1)| = 1$. In particular, the fixed point 0 is neutral (neither attracting nor repelling).

D) One of the assumptions of the Contraction Mapping Theorem was that $|F'(x)|$ is bounded by a number $\lambda < 1$. Here this is not the case and we cannot use the estimates of the theorem directly. Nevertheless, the fixed point is unique and convergence takes place for every $x_0$ in $[0, 1]$.

E) The order of convergence is $p = 1$, the rate is $C = 1$. This follows by writing

$$\frac{|x_{n+1} - 0|}{|x_n - 0|} = 1 - x_n \to 1, \quad n \to \infty.$$