Numerical Analysis
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**BISECTION**

Let $f(x)$ be a function continuous on the interval $[a, b]$ and such that the values $f(a)$ and $f(b)$ are nonzero and have opposite signs. For instance,

$$f(a) < 0, \quad f(b) > 0.$$  

Then by the Intermediate Value theorem, the equation $f(x) = 0$ necessarily has a solution in $(a, b)$. Call it $\alpha$ and suppose that $f(x)$ has no other roots in $[a, b]$.

Let $x_0 = a$ (or $x_0 = a$) and let $x_1 = \frac{1}{2}(a + b)$. If $f(x_1)$ is zero or is sufficiently small, we can output $x_1$ as the exact or an approximate value of $\alpha$. If, in addition, $f(x)$ is differentiable at $\alpha$, the approximation

$$|f(x_1)| \approx |f'(\alpha)||x_1 - \alpha|,$$

gives some information on how close $x_1$ is to $\alpha$. However, since $\alpha$ is unknown and since $f'(\alpha)$ may not even exist, all we can generally say is that

$$|x_1 - \alpha| \leq \frac{1}{2} (b - a).$$

Note that this error bound makes no reference to the function $f(x)$.

If $f(x_1)$ is too far from zero, we can replace $[a, b]$ by either $[a, x_1]$ or by $[x_1, b]$, depending on the sign of $f(x_1)$, and let $x_2$ be the midpoint of the new interval. Then

$$|x_2 - \alpha| \leq \frac{1}{4} (b - a).$$

Continuing in this fashion, we construct a sequence of nested closed intervals each of which contains $\alpha$ and whose midpoints $x_n$ satisfy

$$|x_n - \alpha| \leq \frac{1}{2^n} (b - a).$$

In particular, points $x_n$ converge to $\alpha$. 

How many steps are needed to meet a given accuracy? A simple estimate gives:

\[
\frac{1}{2^n} (b - a) < \varepsilon
\]

\[
2^n > \frac{b - a}{\varepsilon}
\]

\[
n > \log_2 \frac{b - a}{\varepsilon}.
\]

It thus takes at most \(O(\ln \frac{1}{\varepsilon})\) steps to complete a bisection process.

With some luck, the process would quickly produce an exact or an almost exact answer. The best possible outcome would be \(x_n = \alpha\) for a not so large integer \(n\) (\(x_0 = \alpha\) is, of course, ideal). But what is the worst possible scenario?

Example. Let’s apply bisection to the equation \(x - \frac{1}{3} = 0\) on the interval \([0, 1]\). Set \(x_0 = 0\). Then \(x_1 = \frac{1}{2} \), \(x_2 = \frac{1}{4} \), \(x_3 = \frac{3}{8} \), \(x_4 = \frac{5}{16} \), ... , and \(|x_0 - \frac{1}{3}| = \frac{1}{3} \), \(|x_1 - \frac{1}{3}| = \frac{1}{6} \), \(|x_2 - \frac{1}{3}| = \frac{1}{12} \), \(|x_3 - \frac{1}{3}| = \frac{1}{24} \), \(|x_4 - \frac{1}{3}| = \frac{1}{48} \), ... . The absolute error \(|x_n - \frac{1}{3}|\) is halved at each step.

Example. Consider now \(x - \frac{1}{7} = 0\) on \([0, 1]\). If \(x_0 = 0\), then \(x_1 = \frac{1}{2} \), \(x_2 = \frac{1}{4} \), \(x_3 = \frac{1}{8} \), \(x_4 = \frac{3}{16} \), ... , and \(|x_0 - \frac{1}{7}| = \frac{1}{7} \), \(|x_1 - \frac{1}{7}| = \frac{5}{14} \), \(|x_2 - \frac{1}{7}| = \frac{3}{28} \), \(|x_3 - \frac{1}{7}| = \frac{1}{56} \), \(|x_4 - \frac{1}{7}| = \frac{5}{112} \), ... . The absolute error \(|x_n - \frac{1}{7}|\) is not monotone and the successive ratios \(\frac{|x_{n+1} - \frac{1}{7}|}{|x_n - \frac{1}{7}|}\) form a periodic sequence: \(\frac{5}{2}, \frac{3}{10}, \frac{1}{6}, \frac{5}{2}, \frac{3}{10}, \frac{1}{6}, \ldots \).

In fact, if \(\alpha - a\) and \(b - \alpha\) are both rational then either the process \(x_0 = a, x_1 = \frac{1}{2}(a + b), \ldots \) terminates after finitely many steps or the sequence of ratios \(\frac{|x_{n+1} - \alpha|}{|x_n - \alpha|}\) is eventually periodic.

The bound \(|x_n - \alpha| \leq \frac{1}{2^n} (b - a)\) ensures that convergence is always at least linear and has a rate of at most \(\frac{1}{2}\). On the other hand, the process may be arbitrarily irregular, in the sense that, as \(n \to \infty\), \(|x_{n+1} - \alpha|\) may not be controlled by \(C|x_n - \alpha|\) for any fixed \(C\).

A simple geometric argument (try to find it) gives the bound

\[
|x_{n+1} - \alpha| \leq \frac{1}{2} \max \left\{ |x_n - \alpha|, |x_{n-1} - \alpha| \right\}.
\]

Thus, on average, \(|x_{n+1} - \alpha|\) is controlled by \(\frac{1}{2}\) times \(|x_n - \alpha|\), which agrees with our intuitive understanding and terminology.