Approximation of irrational numbers.

Let \( \alpha \) be an irrational number. Its decimal representation is then nonterminating and nonrepeating:

\[
\alpha = A.a_1a_2a_3a_4a_5\ldots,
\]

where \( A \) is the integral part of \( \alpha \). For instance,

\[
\sqrt{2} = 1.414235623730950488\ldots.
\]

To approximate \( \sqrt{2} \), we may take truncated decimal expansions as follows:

\[
\begin{align*}
1 \\
1.4 \\
1.41 \\
1.414 \\
1.4142 \\
1.41423 \\
\ldots.
\end{align*}
\]

All numbers in the preceding sequence are rational and they converge to \( \sqrt{2} \) from below. In fact, the difference between \( \sqrt{2} \) and its \( n \)-th rational approximation is smaller than \( 10^{1-n} \). To achieve a good approximation, one needs to choose \( n \) sufficiently large:

\[
\sqrt{2} \approx \frac{1414235623730950488}{1000000000000000000}.
\]

Could a good approximation of \( \sqrt{2} \) have a not so large denominator? Indeed, if possible, we would like to operate with smaller integers. Take a look at the following example.

**EXAMPLE**  Among all rational fractions with denominator 12, the one closest to \( \sqrt{2} \) is \( \frac{17}{12} \),

\[
\sqrt{2} - \frac{17}{12} \approx .00245.
\]

At the same time,

\[
\sqrt{2} - \frac{141}{100} \approx .0042.
\]

Obviously, \( \frac{17}{12} \) is a better approximation than 1.41, and 12 is a lot smaller than 100.

Let’s introduce a measure of our approximation of \( \alpha \) by a rational number \( p/q \), the approximation coefficient:

\[
\kappa = q \left| \alpha - \frac{p}{q} \right| = |\alpha q - p|.
\]

If \( \alpha - \frac{p}{q} \) is small and \( q \) is not so large, then the coefficient \( \kappa \) is small. Conversely, if \( \kappa \) is small, then \( \alpha - \frac{p}{q} \) is even smaller. Thus the size of \( \kappa \) tells us about the quality of our approximation.
EXAMPLE  Out of three approximations, 3/2, 7/5, and 1.41, to $\sqrt{2}$, which one has the smallest $\kappa$? A quick computation gives the answer:

$$|2\sqrt{2} - 3| \approx .17$$
$$|5\sqrt{2} - 7| \approx .07 \quad *$$
$$|100\sqrt{2} - 141| \approx .42 .$$

Given the size of its denominator, 7/5 is a very good choice. But 17/12 is, of course, better than all three.

How to find these good approximations? Do they always exist? The answer has to do with continued fractions. We’ll only look at a partial explanation here.

THEOREM  Let $\alpha$ be a given irrational number, and let $N$ be any positive integer. Then there exists a rational fraction $p/q$ such that

$$|\alpha q - p| < \frac{1}{N} \quad \text{and} \quad q \leq N.$$ 

In particular, $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$. 

The preceding fact has an interesting illustration. Consider the family of horizontal lines $y = q$, where $q = 0, 1, 2, 3, \ldots$, and the family of parallel lines $x = \alpha y - p$, where $p = 0, \pm 1, \pm 2, \pm 3, \ldots$. The two families intersect in a lattice of points with components $(\alpha q, p)$. For instance, the line $y = 3$ meets the line $x = \alpha y - 5$ at $(3\alpha - 5, 3)$.

To each rational fraction $p/q$, associate the point $(\alpha q, p)$, whose $x$-component has absolute value $\kappa$ and whose $y$-component is the denominator $q$. Now draw a rectangle with vertices at $(-\frac{1}{N}, 0)$, $(\frac{1}{N}, 0)$, $(-\frac{1}{N}, N + 1)$, $(\frac{1}{N}, N + 1)$. The theorem says that for any choice of $N$, the rectangle must contain a lattice point in its interior. Even if $N$ is very very large, i.e. the rectangle is very very thin.

Plot and experiment for better understanding. If you carefully examine this geometric construction, you may see a way to prove the theorem.