Order and Rate

Let \( \{a_n\} \) be a sequence of positive numbers converging to 0. We would like to measure how fast the terms \( a_n \) decay.

The situation that we have in mind is that of the error sequence \( a_n = |x_n - \alpha| \) of an approximation process \( x_n \to \alpha \). Knowing the speed of decay would help us, given a tolerance \( \varepsilon \), to determine a number of steps sufficient to have \( |x_n - \alpha| < \varepsilon \).

There are various ways to measure the speed of decay, and sometimes the same term carries different meanings. To avoid ambiguity, we will agree on definitions.

Assume that for some numbers \( p \geq 1 \) and \( C > 0 \) the condition

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n^{p}} = C
\]

is satisfied. In this case we will say that \( \{a_n\} \) has the order of convergence \( p \). The limit value \( C \) is called the rate of convergence or the asymptotic constant.

Informally, (1) says that

\[
a_{n+1} \approx C a_n^p
\]

for large \( n \), but it is not always the case that the limit exists.

It is easy to see that larger values of \( p \), and smaller values of \( C \) for the same \( p \), correspond to faster convergence.

EXAMPLE. For \( a_n = \frac{1}{n} \), we have \( p = 1 \) and \( C = 1 \) since \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \).

In fact,

\[
\frac{a_{n+1}}{a_n^{q}} = \frac{n^q}{n+1} \to \begin{cases} 
0, & q < 1 \\
1, & q = 1 \\
\infty, & q > 1.
\end{cases}
\]

EXAMPLE. For \( a_n = 2^{-n} \), we have \( p = 1 \) and \( C = \frac{1}{2} \), because \( a_{n+1} = \frac{1}{2} a_n \). In fact,

\[
\frac{a_{n+1}}{a_n^{q}} = 2^{q-n-1} \to \begin{cases} 
0, & q < 1 \\
\frac{1}{2}, & q = 1 \\
\infty, & q > 1.
\end{cases}
\]
The choice of $p$ in (1) may only seem arbitrary. It is easy to check that if the limit superior $\lim_{n \to \infty} \frac{a_{n+1}}{a_n^p}$ is finite then for any $q < p$ one has

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n^q} \leq \lim_{n \to \infty} \frac{a_{n+1}}{a_n^p} = 0.$$ 

Similarly, if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n^p}$ is positive then for any $q > p$ one has

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n^q} \geq \lim_{n \to \infty} \frac{a_{n+1}}{a_n^p} / \lim_{n \to \infty} a_n^{q-p} = \infty.$$ 

Therefore there is a unique exponent $p \geq 0$ such that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n^q} = \begin{cases} 
0, & q < p \\
C, & q = p \\
\infty, & q > p.
\end{cases}$$

In particular, if the limit of $a_{n+1}/a_n^p$ exists and equals $0 < C < \infty$ then $p$ is the order and $C$ is the rate.

In many cases, the sequence $a_n$ arising as or dominating an error sequence is regular, in the sense that $\frac{a_{n+1}}{a_n^p}$ converges to some $C > 0$ for some $p \geq 1$. In general, $\frac{a_{n+1}}{a_n^p}$ need not have a positive limit even if $a_n$ are monotone. Irregular behavior is typical for “slow-fast” sequences.

EXAMPLE. Let $a_{2k} = \frac{1}{2^k}$ and $a_{2k+1} = \frac{1}{2^k + 1}$, $k \geq 0$. Then $\frac{a_{n+1}}{a_n^2}$ does not have a limit because $\frac{a_{2k+1}}{a_{2k}} = \frac{2^k}{2^{k+1}} \to 1$ and $\frac{a_{2k}}{a_{2k-1}} = \frac{2^{k-1} + 1}{2^k} \to \frac{1}{2}$.

EXAMPLE. Let $a_{2k} = \frac{1}{\log k}$ and $a_{2k+1} = \frac{1}{k}$, $k \geq 1$. Then $\frac{a_{2k+1}}{a_{2k}} = \frac{\log k}{k}$ and $\frac{a_{2k}}{a_{2k-1}} = \frac{(k-1)p}{\log k}$, implying that no $p$ yields (1) with $0 < C < \infty$.

Variable speed of convergence, as in two preceding examples, affects the behavior of ratios $\frac{a_{n+1}}{a_n^p}$ by dispersing the limit values. This reveals a great deal of inflexibility of our definition. For instance, both sequences $\frac{1}{\log n}$ and $\frac{1}{n}$ have order 1, but their combination does not.

The truth of the matter is that the order/rate scale is not sufficiently fine to differentiate between some sequences that are clearly not in the same category. But rather than using more delicate ways of measuring the speed of decay, we will stick with two basic convergence parameters, order and rate, reinforcing them with one more definition.

We will say that a positive sequence $\{\varepsilon_n\}$ has an order of at least $p$ and a rate of at most $C$ if there is a majorizing sequence $\{a_n\}$,

$$\varepsilon_n \leq a_n,$$

that has exact order $p$ and rate $C$. This will allow us to deal with sequences for which (1) does not apply.
LINEAR CONVERGENCE

Some additional terminology is common when $p = 1$. Let a positive sequence $\{a_n\}$ converge to 0 and satisfy the condition

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = C$$

for some $C \geq 0$.

**LEMMA.** (2) implies $C \leq 1$.

**Proof.** Since $a_n \to 0$, there exist infinitely many indices $n$ such that $a_{n+1}/a_n \leq 1$. Hence $C \leq 1$. \qed

The case $p = 1$ and $0 < C < 1$ is termed *linear convergence*. A linearly convergent sequence ultimately behaves like a geometric sequence with ratio $C$ and its logarithm behaves like a linear function of $n$,

$$\log a_n \sim n \log C + d.$$

The case $p = 1$ and $C = 1$ is termed *sublinear convergence*. This category features sequences that converge intolerably slowly, like

$$a_n = 1/\log \log \log n.$$

For *superlinear convergence*, either $p = 1$ and $C = 0$ or $p > 1$. For instance, $a_n = n^{-n}$ has $p = 1$ and $C = 0$.

Given an error sequence $|x_n - \alpha|$, we say that it converges at least linearly, at least sublinearly, or at least superlinearly if there is a sequence $a_n$, with

$$a_n \geq |x_n - \alpha|,$$

that converges respectively linearly, sublinearly, or superlinearly, i.e., if there is a corresponding majorant. Superlinear convergence (quadratic, cubic, etc) is regarded as fast and desirable, while sublinear convergence is usually impractical.
ADDENDUM

Here are several propositions (proofs skipped) that help us to compute the order and rate. As before, \( \{a_n\} \) is a positive sequence converging to 0.

**PROPOSITION.** If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) is finite and positive, then
\[
\lim_{n \to \infty} \frac{\log a_{n+1}}{\log a_n} = p.
\]
The converse is not true (consider \( a_n = n^{-n} \)).

**PROPOSITION.** If \( \lim_{n \to \infty} \frac{\log a_{n+1}}{\log a_n} = p \), then
\[
\lim_{n \to \infty} n^{\frac{n}{\sqrt{\log a_n}}} = p.
\]
The converse is not true (consider \( a_{2k} = e^{-2k} \) and \( a_{2k+1} = e^{-2k-1} \)).

**PROPOSITION.** The limit inferior
\[
\lim_{n \to \infty} n^{\frac{n}{\sqrt{\log a_n}}} \geq 1.
\]
In particular, the order is always at least 1.

**PROPOSITION.** If \( p = 1 \), then
\[
C = \lim_{n \to \infty} \sqrt[n]{a_n}.
\]

**THREE EXAMPLES**

Let \( a_n \) be positive numbers converging to 0. Determine the order and rate of convergence of the sequence if

- a) \( a_{n+1} = a_n^2 \)
- b) \( a_{n+1} = a_n(a_n - 1) \)
- c) \( a_n = n^{-n} \).

**SOLUTION.** The first two cases are straightforward:

- a) \( a_{n+1}/a_n^2 = 1 \), so \( p = 2 \) and \( C = 1 \) (quadratic convergence).
- b) \( a_{n+1}/a_n^1 = 1 - a_n \to 1 \), so \( p = 1 \) and \( C = 1 \) (sublinear convergence).

In the third case we have

- c) \( p = 1 \) and \( C = 0 \) (superlinear convergence). To show this, argue that \( a_{n+1}/a_n^p \) converges only if \( p \leq 1 \), in which case the limit is 0.