Euler-Maclaurin formula. The error of the trapezoidal sum cancels out completely. This is confirmed by the

\[ M_\delta \text{subintervals (length } \delta \text{).} \]

For instance, \( \ln 2 - R_n < \frac{1}{4n} < L_n - \ln 2 < \frac{1}{2n} \).

\[ M_n = \sum_{k=1}^{n} \frac{1}{n+k} - \frac{1}{n+k+1} \]

f) \( M_n = \sum_{k=1}^{n} \frac{1}{n+k} - \frac{1}{n+k+1} \).

g) \[ |\ln 2 - M_n| < \frac{b-a}{24} \max_{[a,b]} |f''(x)| \delta^2 < \frac{1}{12n\pi}. \]

In fact, \( 0 < \ln 2 - M_n < \frac{1}{24n^2}. \)

2. Let \( f(x) \) be a cubic. Simpson’s method requires an even number of equal subintervals (length \( \delta \)). Consider any 3 consecutive nodes \( x_{m-1}, x_m, x_{m+1} \) and let \( q_m(x) \) be the quadratic polynomial that agrees with \( f(x) \) at these nodes. Then \( f(x) - q_m(x) \) is a cubic polynomial with roots \( x_{m-1}, x_m, x_{m+1} \). Write \( f(x) - q_m(x) \) as \( u(h) = Ah^3 + Bh^2 + Ch + D \), where \( h = x - x_m \). Then

\[ u(0) = u(h) = u(-h) = 0, \]

which implies that \( B = D = 0 \). Hence \( u(h) \) is odd and

\[ \int_{x_{m-1}}^{x_{m+1}} (f(x) - q_m(x)) \, dx = \int_{-\delta}^{\delta} (Ah^3 + Ch) \, dh = 0, \]

i.e., the error of \( \int_{x_{m-1}}^{x_{m+1}} q_m(x) \, dx \) is zero for each \( m \).

Quiz 1

A) \( L_3 = \frac{1}{3} \ln \frac{20}{9}, R_3 = \frac{1}{3} \ln \frac{40}{9}. \)
B) \( L_3 < I < R_3 \) because \( \ln x \) is increasing.
C) \( I - L_3 > R_3 - I \) because \( \ln x \) is increasing and concave down.
D) \( M_3 = \frac{1}{3} \ln \frac{77}{31}. \)
E) \( M_3 > I \) because \( (\ln x)' = 1/x \) is decreasing.

Homework 2

\[ I = \int_0^{2\pi} \cos x \, dx = 0 \text{ and } \]

\[ T_n = \frac{2\pi}{n} \left( \frac{1}{2} + \sum_{k=1}^{n-1} \cos\left( \frac{2\pi}{n} k \right) \right) \]

\[ = \frac{2\pi}{n} \sum_{k=0}^{n-1} \cos\left( \frac{2\pi}{n} k \right) = 0. \]

Indeed, the sum of \( n > 1 \) vectors pointing to the vertices of a regular \( n \)-gon from 0 to 2\( \pi \), the center is the zero vector, and hence their \( x \)-components \( \cos(\frac{2\pi}{n} k) \) sum up to 0.

Thus \( I = T_n \) for every \( n > 1 \); due to symmetry and periodicity of \( \cos x \) on \([0, 2\pi]\), the error of the trapezoidal sum cancels out completely. This is confirmed by the Euler-Maclaurin formula.