Exercises 5

1. $3a^2 = \frac{(a+h)^3 - a^3 - h^3}{h}$. This implies that $c = a + \frac{1}{3}h$.

2. $4a^3 = \frac{(a+h)^4 - (a-h)^4}{2h^2}$. This implies that $c = a$.

3. Let $h = 0.1$ and write $p(x) = f(-h)\frac{x(x-2h)}{6h^2} + f(0)\frac{x(x+h)(x-2h)}{2h^3} + h^2f(0)^3$. Then $p'(0) = \frac{-4f(-h)+3f(0)+f(2h)}{6h} = \frac{1}{h}(-3 \cdot 4 + 3 \cdot 2.8 + 2.7) = -1.5$.

4. Let $f''(a) \approx Af(a) + Bf(a) + Cf(a + 2h)$ be exact for $f(x) = x, x^2$. Then

   $$0 = A + B + C$$
   $$0 = A(a - h) + Ba + C(a + 2h)$$
   $$2 = A(a - h)^2 + Ba^2 + C(a + 2h)^2.$$  

This gives $A = \frac{2}{3h^2}, B = -\frac{1}{h^2}$, and $C = \frac{1}{3h^2}$. Hence

$$f''(a) \approx \frac{2f(a - h) - 3f(a) + f(a + 2h)}{3h^2}.$$  

Remark. The error of the right-hand side can be found by combining

$$f(a - h) = f(a) - f'(a)h + \frac{1}{2}f''(a)h^2 - \frac{1}{6}f'''(c_1)h^3$$
$$f(a) = f(a)$$

$$f(a + 2h) = f(a) + 2f'(a)h + 2f''(a)h^2 + \frac{4}{3}f'''(c_2)h^3.$$  

Indeed, with the above choice of $A, B, C$,

$$Af(a - h) + Bf(a) + Cf(a + 2h) = f''(a) + (\frac{4}{3}Cf'''(c_2) - \frac{1}{6}Af'''(c_1))h^3$$
$$= f''(a) + \frac{1}{6}(Af'''(c_2) - f'''(c_1))h.$$  

Hence the absolute error is bounded by $\frac{h^3}{6} \max_{|a-h,a+2h|} |f'''(x)|h$.

Quiz 2

Assume that $f(x)$ is three times continuously differentiable. Combining

$$f(a + h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \frac{h^3}{6}f'''(c_1)h^3$$
$$f(a) = f(a)$$

$$f(a - h) = f(a) - f'(a)h + \frac{1}{2}f''(a)h^2 - \frac{1}{6}f'''(c_2)h^3$$

we obtain

$$Af(a-h)+Bf(a)+Cf(a-h) = (A+B+C)f(a)h(A-C)f'(a)+\frac{h^2}{2}(A+C)f''(a)+\frac{h^3}{6}(A+C)A'''(c_1)-Cf'''(c_2)).$$  

Coefficient conditions $A + B + C = 0, h(A - C) = 1, \frac{h^2}{2}(A + C) = 0$ give $A = -C = \frac{1}{2h}$ and $B = 0$, and so

$$f'(a) = \frac{f(a + h) - f(a - h)}{2h} - \frac{h^2}{6}(\frac{1}{2}f'''(c_1) + \frac{1}{2}f'''(c_2))$$

$$= \underbrace{\frac{f(a + h) - f(a - h)}{2h}}_{\text{error}} - \frac{h^2}{6} \underbrace{f'''(c)}_{\text{error}}.$$
Homework 4

Let $f(x) = kx^4 + \ldots$ and suppose that $k \neq 0$ is known. Let $y_0 = f(a)$ and $y_{\pm} = f(a \pm h)$, and let $\tilde{y}_0 = y_0 - \varepsilon_0$ and $\tilde{y}_{\pm} = y_{\pm} - \varepsilon_{\pm}$ be approximations.

Recall that the error term in the second difference formula was found as follows. Assuming that $f(x)$ is four times continuously differentiable, combine

$$f(a + h) = f(a) + f'(a)h + \frac{1}{2} f''(a)h^2 + \frac{1}{6} f'''(a_1)h^3 + \frac{1}{24} f''''(a_2)h^4$$

$$f(a) = f(a)$$

$$f(a - h) = f(a) - f'(a)h + \frac{1}{2} f''(a)h^2 - \frac{1}{6} f'''(a)h^3 + \frac{1}{24} f''''(a_2)h^4$$

to obtain

$$Af(a-h) + Bf(a) + Cf(a+h) = (A + B + C)f(a) + h(A - C)f'(a) + \frac{h^2}{2} (A + C) f''(a) + \frac{h^3}{3!} (A - C) f'''(a) + \frac{h^4}{4!} (A f''''(a_1) + C f''''(a_2)).$$

Coefficient conditions $A + B + C = 0$, $h(A - C) = 0$, $\frac{h^2}{2} (A + C) = 1$ give $A = C = \frac{1}{h^2}$ and $B = -\frac{1}{h^2}$. Hence

$$f''(a) = \frac{f(a + h) - 2f(a) + f(a - h)}{h^2} - \frac{h^2}{12} \left( \frac{1}{2} f'''(a_2) + \frac{1}{2} f'''(a_1) \right).$$

We then have

$$f''(a) - \frac{\tilde{y}_{-2\tilde{y}_0 + \tilde{y}_+}}{h^2} = f''(a) - \frac{y_{-2y_0 + y_+}}{h^2} + \frac{\varepsilon_{-2\varepsilon_0 + \varepsilon_+}}{h^2} = -\frac{h^2}{12} f^{(4)}(c) + \frac{\varepsilon_{-2\varepsilon_0 + \varepsilon_+}}{h^2} = -2kh^2 + \frac{\varepsilon_{-2\varepsilon_0 + \varepsilon_+}}{h^2},$$

where $\varepsilon = \varepsilon_0 - 2\varepsilon_0 + \varepsilon_+$. This gives a bound on the absolute error:

$$\left| f''(a) - \frac{\tilde{y}_{-2\tilde{y}_0 + \tilde{y}_+}}{h^2} \right| \leq 2|k|h^2 + \frac{1}{12} h^2.$$

As a function of $h$, the right-hand side has a unique minimum of $2\sqrt{|k|}$ at $h = \sqrt{\frac{1}{2} \frac{1}{|k|}}$. As a function of $|\varepsilon|$, the right-hand side is monotone increasing. Since the absolute errors $|\varepsilon_0|, |\varepsilon_{\pm}|$ are given not to exceed .1, the worst case scenario occurs when $|\varepsilon| = .1$. So

$$h = \frac{1}{\sqrt{5} |k|}$$

is a reasonable choice for the stepsize.

If the value of $k$ is not known, it can be estimated using the approximate values of the function. In this case the question is more involved.