Order and Rate

Let \(\{a_n\}\) be a sequence of positive numbers converging to 0. We would like to measure how fast the terms \(a_n\) decay. This is of interest for the error sequence \(a_n = |x_n - \alpha|\) of an approximation process \(x_n \to \alpha\). Knowing the speed of decay would help us, given a tolerance \(\varepsilon\), to determine a sufficient number of steps to have \(|x_n - \alpha| < \varepsilon\).

There are various ways to measure the speed of decay, and sometimes the same term carries different meanings. To avoid ambiguity, we should agree on definitions.

We will say that \(\{a_n\}\) has the order of convergence \(p\) if the condition

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n^p} = C
\]

is satisfied and \(0 < C < \infty\). The limit value \(C\) is called the rate of convergence or the asymptotic constant. Informally, (1) says that

\[
a_{n+1} \approx C a_n^p
\]

for large \(n\), but it is not always the case that the limit exists.

Larger values of \(p\), and smaller values of \(C\) for the same \(p\), correspond to faster convergence.

EXAMPLE. For \(a_n = \frac{1}{n}\), we have \(p = 1\) and \(C = 1\) since \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1\).

In fact,

\[
\frac{a_{n+1}}{a_n^q} = \frac{n^q}{n+1} \to \begin{cases} 0, & q < 1 \\ 1, & q = 1 \\ \infty, & q > 1. \end{cases}
\]

EXAMPLE. For \(a_n = 2^{-n}\), we have \(p = 1\) and \(C = \frac{1}{2}\), because \(a_{n+1} = \frac{1}{2} a_n\). In fact,

\[
\frac{a_{n+1}}{a_n^q} = 2^{q-n-1} \to \begin{cases} 0, & q < 1 \\ \frac{1}{2}, & q = 1 \\ \infty, & q > 1. \end{cases}
\]

The choice of \(p\) in (1) may only seem arbitrary. It is easy to check that if \(\lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) is finite then for any \(q < p\) one has

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n^q} \leq \lim_{n \to \infty} \frac{a_{n+1}}{a_n^p} \lim_{n \to \infty} a_n^{p-q} = 0.
\]
Similarly, if \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \) is positive then for any \( q > p \) one has
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n^q} \geq \lim_{n \to \infty} \frac{a_{n+1}}{a_n^p} / \lim_{n \to \infty} a_n^{q-p} = \infty.
\]
Therefore there is a unique exponent \( p \geq 0 \) such that
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n^q} = \begin{cases} 
0, & q < p \\
C, & q = p \\
\infty, & q > p.
\end{cases}
\]
In particular, if the limit of \( a_{n+1}/a_n^p \) exists and equals \( 0 < C < \infty \) then \( p \) is the order and \( C \) is the rate.

For any \( p \geq 1 \) it is easy to construct a sequence of order \( p \).

EXAMPLE. For any \( p > 1 \) and \( M > 0 \), the sequence \( a_n = Me^{-p^n} \) has order \( p \) and rate \( M^{1-p} \).

In many cases, the sequence \( a_n \) arising as (or majorizing) an error sequence is regular, in the sense that \( \frac{a_{n+1}}{a_n^p} \) converges to some \( C > 0 \) for some \( p \geq 1 \).

In general, \( \frac{a_{n+1}}{a_n^p} \) need not have a positive limit even if \( a_n \) are monotone.

Irregular behavior is typical for combinations of “slow” and “fast” sequences.

EXAMPLE. Let \( a_{2k} = \frac{1}{2^k} \) and \( a_{2k+1} = \frac{1}{2^{k+1}}, \ k \geq 0 \). Then \( \frac{a_{n+1}}{a_n} \) does not have a limit because \( \frac{a_{2k+1}}{a_{2k}} = \frac{2^k}{2^{k+1}} \to 1 \) and \( \frac{a_{2k}}{a_{2k-1}} = \frac{2^{k-1}+1}{2^k} \to \frac{1}{2} \).

EXAMPLE. Let \( a_{2k} = \frac{1}{\log k} \) and \( a_{2k+1} = \frac{1}{k}, \ k \geq 1 \). Then \( \frac{a_{2k+1}}{a_{2k}} = \frac{\log k}{k} \) and \( \frac{a_{2k}}{a_{2k-1}} = \frac{(k-1)p}{\log k} \), implying that no \( p \) yields (1) with \( 0 < C < \infty \).

Variable speed of convergence in two preceding examples affects the behavior of ratios \( \frac{a_{n+1}}{a_n^p} \) by dispersing the limit values of their subsequences. This reveals a great deal of inflexibility of our definition. Indeed, both sequences \( \frac{1}{\log n} \) and \( \frac{1}{n} \) have order 1, but their combination does not.

In fact, the order/rate grid is not sufficiently fine to differentiate between some sequences that are clearly not in the same category. But rather than consider more delicate ways of measuring the speed of decay, we will stick with two basic convergence parameters, order and rate, reinforcing them by one more definition.

We will say that a positive sequence \( \{ \varepsilon_n \} \) has an order of at least \( p \) and a rate of at most \( C \) if there is a majorizing sequence \( \{ a_n \} \),
\[
\varepsilon_n \leq a_n,
\]
that has exact order \( p \) and rate \( C \). This will allow us to deal with sequences for which (1) does not apply.
Linear Convergence

Some additional terminology is common when \( p = 1 \). Let a positive sequence \( \{a_n\} \) converge to 0 and satisfy the condition

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = C
\]

for some \( C \geq 0 \).

**Lemma.** (2) implies \( C \leq 1 \).

**Proof.** Since \( a_n \to 0 \), there exist infinitely many indices \( n \) such that
\[
a_{n+1}/a_n \leq 1.
\]
Hence \( C \leq 1 \). \( \Box \)

The case \( p = 1 \) and \( 0 < C < 1 \) is termed linear convergence. A linearly convergent sequence ultimately behaves like a geometric sequence with ratio \( C \) and its logarithm behaves as a linear function of \( n \),
\[
\log a_n \sim n \log C + d.
\]

The case \( p = 1 \) and \( C = 1 \) is termed sublinear convergence. This category features sequences that converge intolerably slowly, like
\[
a_n = 1/\log \log \log n.
\]

For superlinear convergence, either \( p = 1 \) and \( C = 0 \) or \( p > 1 \). For instance, \( a_n = n^{-n} \) has \( p = 1 \) and \( C = 0 \).

Given an error sequence \( |x_n - \alpha| \), we say that it converges at least linearly, at least sublinearly, or at least superlinearly if there is a sequence \( a_n \), with
\[
a_n \geq |x_n - \alpha|,
\]
that converges respectively linearly, sublinearly, or superlinearly, i.e., if there is a corresponding majorant. Superlinear convergence (quadratic, cubic, etc) is regarded as fast and desirable, while sublinear convergence is regarded as impractical.
ADDENDUM

Here are several propositions (proofs skipped) that help to compute the order and rate. As before, \( \{a_n\} \) is a positive sequence converging to 0.

PROPOSITION. Let \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = C \) be finite and positive. Then
\[
\lim_{n \to \infty} \frac{\log a_{n+1}}{\log a_n} = p.
\]
The converse is not true (consider \( a_n = n^{-n} \)).

PROPOSITION. Let \( \lim_{n \to \infty} \frac{\log a_{n+1}}{\log a_n} \) exist and equal \( p \). Then
\[
\lim_{n \to \infty} n \sqrt{|\log a_n|} = p.
\]
The converse is not true (consider \( a_{2k} = e^{-2k} \) and \( a_{2k+1} = e^{-2k-1} \)).

PROPOSITION.
\[
\lim_{n \to \infty} \sqrt{|\log a_n|} \geq 1.
\]
In particular, the order is always at least 1.

PROPOSITION. If \( p = 1 \) then
\[
\log C = \lim_{n \to \infty} \frac{\log a_n}{n}.
\]