Complex Analysis
Grinshpan

CAUCHY-HADAMARD FORMULA

Theorem [Cauchy, 1821] The radius of convergence of the power series \( \sum_{n=0}^{\infty} c_n (z - z_0)^n \) is

\[
R = \frac{1}{\lim_{n \to \infty} \sqrt[n]{|c_n|}}
\]

Example. For any increasing sequence of natural numbers \( n_j \) the radius of convergence of the power series \( \sum_{j=1}^{\infty} z^{n_j} \) is \( R = 1 \).

Proof. Let \( R = 1/ \lim_{n \to \infty} \sqrt[n]{|c_n|} \in [0, \infty] \).

If \( R < \infty \), choose any \( r > R \). Then \( \lim_{n \to \infty} \sqrt[n]{|c_n|} > 1/r \) and so \( |c_n| r^n > 1 \) for infinitely many indices \( n \). So \( c_n (z - z_0)^n \) does not approach 0 for any \( z \) with \( |z - z_0| = r > R \). So the power series diverges for any \( z \) with \( |z - z_0| > R \).

If \( R > 0 \), choose any \( 0 < r < R \). Then \( \lim_{n \to \infty} \sqrt[n]{|c_n|} < 1/r \) and so \( |c_n| r^n < 1 \) for all but finitely many indices \( n \). Hence for any \( z \) with \( |z - z_0| < r \),

\[
\sum_{n=0}^{\infty} |c_n| |z - z_0|^n = \sum_{n=0}^{\infty} |c_n| r^n \left| \frac{z - z_0}{r} \right|^n \leq M \sum_{n=0}^{\infty} \left| \frac{z - z_0}{r} \right|^n < \infty.
\]

So the power series converges for any \( z \) with \( |z - z_0| < R \).

It follows that the radius of convergence is \( R \). \( \square \)

Exercises. Find the radius of convergence of each of the following power series.

\[
\sum_{n=1}^{\infty} (1 + 1/n)^n z^n
\]

\[
\sum_{n=1}^{\infty} (4 + i/n)^n z^{2n}
\]

\[
\sum_{n=1}^{\infty} \frac{z^n}{n^p}
\]

\[
\sum_{n=0}^{\infty} \frac{z^n}{n!}
\]

\[
\sum_{n=0}^{\infty} (2 + i^n)^n z^n
\]

\[
\sum_{n=1}^{\infty} \sin(n) z^n
\]