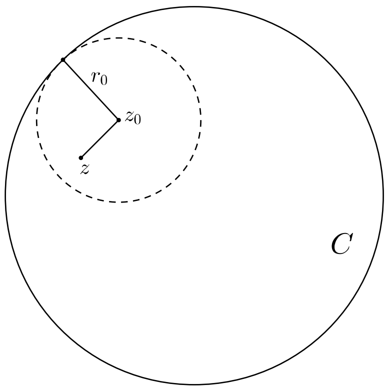
Cauchy integral and local series representation

Let $f(z)$ be holomorphic in an open set G, and let a circle C be contained in G together with its interior. Then, by the Cauchy formula for a circle,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where C is oriented counterclockwise and z is in the interior of C. Fix z_0 in the interior of C. It will be shown that f has a power series expansion with center z_0.



Let r_0 be the distance from z_0 to C. Then, for each z with $|z - z_0| < r_0$, and any ζ on C, we have a geometric series expansion

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\,\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

Hence

$$\frac{f(\zeta)}{\zeta - z} = \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,(z - z_0)^n, \qquad \zeta \in C, \quad |z - z_0| < r_0.$$

For each z with $|z - z_0| < r_0$ and each $0 < r < 1$, the power series on the right converges uniformly in ζ in the region

$$\left|\frac{z - z_0}{\zeta - z_0}\right| \le r.$$

Integrating counterclockwise over C, we find that

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \qquad |z - z_0| < r_0,$$

where

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta, \qquad n = 0, 1, 2, \ldots .$$

Thus at each point $z \in G$, f has a power series representation and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \qquad n = 0, 1, 2, \ldots .$$