Lebesgue’s Criterion for Riemann Integrability

**Theorem** [Lebesgue, 1901]: A bounded function on a closed bounded interval is Riemann-integrable if and only if the set of its discontinuities is a null set.

Let us reformulate the theorem. Suppose that \( f \) is a bounded function on \([a, b]\) and \( D \) is the set of its discontinuities. Recall that we can write \( D \) as a countable union,

\[
D = \bigcup_{n=1}^{\infty} D^{1/n},
\]

of closed and bounded sets \( D^{1/n} \). Thus \( D \) is a null set if and only if each \( D^{1/n} \) is a null set, and \( D^{1/n} \) is a null set if and only if it has outer content zero. So Lebesgue’s condition means each set \( D^{\alpha} \) has outer content zero.

**Proof.** Assume first that \( f \) is Riemann-integrable. Choose any \( \alpha > 0 \). Given \( \varepsilon > 0 \), select a partition \( P \) such that

\[
U(f, P) - L(f, P) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) < \alpha \varepsilon.
\]

Consider the subintervals of the partition that contain points of \( D^{\alpha} \). On any such subinterval \( \sup f - \inf f \geq \alpha \). So the total length of these subintervals is less than \( \varepsilon \). Hence, for every \( \varepsilon > 0 \), the set \( D^{\alpha} \) can be covered by finitely many intervals of total length less than \( \varepsilon \). Therefore \( D^{\alpha} \) has outer content zero.

Conversely, suppose that each \( D^{\alpha} \) has outer content zero. Let \( M \) be a bound for \( f \). Given any \( \varepsilon > 0 \), let \( \alpha = \frac{1}{4} \varepsilon/(b-a) \) and select a cover of \( D^{\alpha} \) by finitely many open subintervals of \([a, b]\) of total length less than \( \frac{1}{4} \varepsilon/M \). On the complement to the union of these subintervals \( f \) is \( \alpha \)-continuous. In fact, as this complement is closed and bounded, \( f \) is uniformly \( \alpha \)-continuous on it. So, refining the complementary intervals if necessary, we can form a partition \( P \) of \([a, b]\) such that

\[
U(f, P) - L(f, P) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1})
= \sum_{\text{covering}} (M_k - m_k)(x_k - x_{k-1}) + \sum_{\text{complement}} (M_k - m_k)(x_k - x_{k-1})
< 2M \frac{\varepsilon}{4M} + \alpha(b-a) = \varepsilon.
\]

So \( f \) is Riemann-integrable. \( \square \)