Taylor approximation: an example

Let us approximate $f(x) = \ln x$ by its Taylor polynomial $T_n(x)$ centered at $x_0 = 1$.

We will aim to satisfy the error bound $|\ln x - T_n(x)| < 0.001$ for $0.5 \le x \le 1.5$.

Recall that $T_n(x) = \displaystyle\sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k$.

The value at $x_0 = 1$ is $f(1) = 0$.

The derivatives $f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)} = -6x^{-4}$, $\ldots$

can be written is follows:

$$f^{(k)}(x) = (-1)^{k-1}(k-1)!x^{-k}.$$

This gives the coefficients:

$$\frac{f^{(k)}(1)}{k!} = \frac{(-1)^{k-1}}{k}, \quad k = 1,\, 2,\, 3,\, \ldots,\, n.$$

So

$$T_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \ldots + \frac{(-1)^{n-1}}{n}\,(x-1)^n.$$

The Lagrange remainder estimate is

$$|\ln x - T_n(x)| \le \frac{M}{(n+1)!}\,|x-1|^{n+1}.$$

Here $0.5 \le x \le 1.5$ and M is the maximum of $|f^{(n+1)}(x)|$ over the interval $[0.5,\, 1.5]$.

Note that $M = |f^{(n+1)}(0.5)| = n!2^{n+1}$ and that $|x-1|^{n+1} \le 1/2^{n+1}$, $0.5 \le x \le 1.5$.

Therefore,

$$|\ln x - T_n(x)| \le \frac{n!2^{n+1}}{(n+1)!}\frac{1}{2^{n+1}} = \frac{1}{n+1}.$$

Thus the desired proximity is achieved by choosing n so that $n+1 > 1000$ or $n = 1000$.

The Taylor polynomial

$$T_{1000}(x) = \sum_{k=1}^{1000} \frac{(-1)^{(k-1)}}{k}(x-1)^k.$$

approximates $\ln x$ on the interval $[0.5,\, 1.5]$ to within 0.001.