ABSOLUTE CONVERGENCE

The series

$$a_1 + a_2 + \ldots + a_n + \ldots$$

is said to converge absolutely if

$$|a_1| + |a_2| + \ldots + |a_n| + \cdots < \infty.$$

A convergent series may or not converge absolutely.
A divergent series cannot converge absolutely.

The sums

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} - \frac{1}{10^2} + \ldots$$

$$-1 - \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{1}{9^2} - \frac{1}{10^2} + \ldots$$

are absolutely convergent, but the sum

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$$

is not absolutely convergent.

If a convergent series does not converge absolutely, it is said to converge conditionally. The alternating harmonic series converges conditionally.

The absolute convergence is not affected by sign changes $\pm a_n$ or by term rearrangements. Moreover, all rearrangements of an a. c. series converge to the same sum. The conditional convergence is more delicate: it is ruined if a_n are replaced by $|a_n|$ (and may be ruined by a sign change or term rearrangement).

Absolute convergence implies convergence:

$$\text{if} \quad \sum_{n=1}^{\infty} |a_n| < \infty \quad \text{then} \quad \sum_{n=1}^{\infty} a_n \quad \text{converges.}$$

This implication cannot be reversed.
It is sometimes easier to prove convergence by verifying absolute convergence.

Examples.
1. $1 - 1 + 1 - 1 + 1 - 1 + \ldots$ diverges (divergence test).
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges (alt. series test). Not absolutely.
3. $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges (alt. series test). Not absolutely.
4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ converges (alt. series test). Absolutely ($\frac{3}{2}$-series).
5. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely (compare to 2-series).
6. $\sum_{n=1}^{\infty} \frac{\cos^3 n}{n^2}$ converges absolutely (compare to 2-series).
7. $\sum_{n=1}^{\infty} \frac{(-1)^n \sin \frac{1}{n}}{n}$ converges absolutely (limit comparison with 2-series).