Problem set 3.1

1. For each matrix find vectors that span its kernel.

\[
\begin{bmatrix}
1 & 2 \\
3 & 4
\end{bmatrix}, \quad \begin{bmatrix}
1 & 2 & 3 \\
0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
2 & 3 \\
1 & 2 & 3
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
1 & 2 \\
1 & 3
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 2 & 0 & 0 & 3 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

2. For each matrix in the preceding problem find vectors that span its image.

3. For each transformation below describe the image and kernel geometrically.
   (a) Reflection about \( y = x/3 \) in \( \mathbb{R}^2 \)
   (b) Orthogonal projection onto the plane \( x + 2y + 3z = 0 \) in \( \mathbb{R}^3 \)
   (c) Counterclockwise rotation through an angle of \( \pi/4 \) in \( \mathbb{R}^2 \)

4. Give an example of a matrix whose image spanned by \[
\begin{bmatrix} 1 \\ 5 \end{bmatrix}.
\]

5. Given an example of a matrix whose image is a plane in \( \mathbb{R}^3 \) normal to \[
\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.
\]

6. Consider a nonzero vector \( v \) in \( \mathbb{R}^3 \). Describe the image and the kernel of the linear transformation \( T(x) = x \cdot v \) (dot product).

7. Consider a nonzero vector \( v \) in \( \mathbb{R}^3 \). Describe the image and the kernel of the linear transformation \( T(x) = x \times v \) (cross product).

8. Let \( A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \). Find the image and the kernel for \( A, A^2, A^3 \).

9. Is the kernel of a matrix necessarily the same as the kernel of its reduced row echelon form?

10. Is the image of a matrix necessarily the same as the image of its reduced row echelon form?

Problem set 3.2

1. Find a nontrivial relation among the vectors \[
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\
3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\]

2. Determine whether the given vectors are linearly dependent.
   (a) \[
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\n3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]
   (b) \[
\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\n3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

3. Let \( u_1, u_2, u_3 \) be mutually perpendicular unit vectors in \( \mathbb{R}^3 \). Argue that these vectors are necessarily linearly independent.