The Cantor pairing function

Let $\mathbb{N}_0 = \{0,\ 1,\ 2,\ \ldots\}$ be the set of nonnegative integers and let $\mathbb{N}_0 \times \mathbb{N}_0$ be the set of all ordered pairs of nonnegative integers.

Consider a function $L(m,n) = am + bn + c$ mapping $\mathbb{N}_0 \times \mathbb{N}_0$ to $\mathbb{N}_0$, not a constant. Observe that $c = L(0,0)$ is necessarily an integer. The same is true of $a = L(1,0) - c$ and $b = L(0,1) - c$. In fact, a and b must be nonnegative integers, not both zero. But then $L(m,n) = L(m-b, n+a)$, for any $m \geq b$. So L cannot be injective. Therefore there does not exist a linear bijection from $\mathbb{N}_0 \times \mathbb{N}_0$ onto $\mathbb{N}_0$.

A quadratic bijection does exist. The so-called Cantor pairing function

$$C(m,n) = \sum_{j=0}^{m+n} j \ + \ m = \tfrac{1}{2}(m+n)(m+n+1) + m,$$

maps $\mathbb{N}_0 \times \mathbb{N}_0$ injectively onto $\mathbb{N}_0$ (Cantor, 1878). If (m,n) is the row-column indexing, $C(m,n)$ gives the following pattern of enumeration:

0	1	3	6	10	15
2	4	7	11	16	
5	8	12	17		
9	13	18			
14	19				
20					

To check that $C(m,n)$ is indeed a bijection, we need the below property.

PROPERTY. If $m+n < m'+n'$, then $C(m,n) < C(m',n')$.

PROOF. Subject to the constraint $m+n=k$, the maximum value of $C(m,n)$ is $C(k,0) = \frac{1}{2}\,k(k+1)+k$ and the minimum value of $C(m,n)$ is $C(0,k) = \frac{1}{2}\,k(k+1)$. If $k' \geq k+1$, then $C(0,k') \geq \frac{1}{2}(k+1)(k+2) = \frac{1}{2}\,k(k+1)+k+1 > C(k,0)$. $\square$

It follows that the equality $C(m,n) = C(m',n')$ implies that $m+n = m'+n'$. But then $m = m'$ and $n = n'$, which gives the injectivity of $C(m,n)$.

Why is $C(m,n)$ surjective? Given any $z \in \mathbb{N}_0$, , let $t_k = \frac{1}{2}\,k(k+1)$ be the largest triangular number not exceeding z. Let $m = z - t_k$ and $n = k - m$. Then $C(m,n) = z$.

No polynomial bijection from $\mathbb{N}_0 \times \mathbb{N}_0$ onto $\mathbb{N}_0$, other than $C(m,n)$ and its reflection $C(n,m)$, has yet been discovered. Fueter and Polya (1923) have shown that no other bijective quadratics exist, and many years later, motivated by computer storage applications, Lew and Rosenberg (1978) have excluded all cubic and quartic polynomials from the consideration. It has been asked repeatedly whether Cantor's pairing is the only bijective polynomial from $\mathbb{N}_0 \times \mathbb{N}_0$ onto $\mathbb{N}_0$. The question is open to this day.