Complex Analysis
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**COMPLEX DIFFERENTIABILITY**

Let $f = u + iv$ be a complex-valued function defined in an open subset $G$ of the complex plane, and let $z_0 = x_0 + iy_0$ be a point of $G$.

**Complex differentiability.** We say that $f(z)$ is differentiable at $z_0$ if there exists

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$ 

Thus $f$ is differentiable at $z_0$ if and only if there is a complex number $c$ such that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0) - c(z - z_0)}{z - z_0} = 0,$$

in which case $f'(z_0) = c$.

**Real differentiability.** We say that $u(x, y)$ is differentiable at $(x_0, y_0)$ if there exist real numbers $a$, $b$ such that

$$\lim_{(x, y) \to (x_0, y_0)} \frac{u(x, y) - u(x_0, y_0) - a(x - x_0) - b(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0,$$

in which case $u_x(x_0, y_0) = a$ and $u_y(x_0, y_0) = b$.

**Complex differentiability vs real.** Writing $c = a + ib$ and noting that

$$\begin{align*}
\text{Re} \left( \frac{f(z) - f(z_0) - c(z - z_0)}{|z - z_0|} \right) &= \frac{u(x, y) - u(x_0, y_0) - a(x - x_0) + b(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}, \\
\text{Im} \left( \frac{f(z) - f(z_0) - c(z - z_0)}{|z - z_0|} \right) &= \frac{v(x, y) - v(x_0, y_0) - b(x - x_0) - a(y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}},
\end{align*}$$

we conclude from the two definitions that $f$ is complex differentiable at $z_0$ if and only if $u$ and $v$ are real differentiable at $(x_0, y_0)$ and the Cauchy–Riemann equations hold at $z_0$,

$$u_x(x_0, y_0) = v_y(x_0, y_0), \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Moreover, when this is the case,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

The equivalence criterion gives a necessary condition for the existence of $f'(z_0)$.

**Necessary condition.** If $f = u + iv$ is differentiable at $z_0$, then $u$ and $v$ have first partial derivatives at $z_0$ and satisfy the Cauchy–Riemann equations at $z_0$.

This condition is only necessary, because the existence of first partial derivatives does not guarantee real differentiability.

**Example.** Let $u(x, y) = xy/(x^2 + y^2)$, for $(x, y) \neq (0, 0)$, and let $u(0, 0) = 0$. Then $u_x(0, 0) = u_y(0, 0) = 0$, but $u(x, y)$ is not differentiable at the origin.
A sufficient condition for the existence of \( f'(z_0) \) comes from Multivariable Calculus: if \( u(x,y) \) has first partial derivatives, then it is differentiable at every point where those partial derivatives are continuous.

**Sufficient condition.** Let the first partial derivatives of \( u \) and \( v \) exist in an open neighborhood of \( z_0 \). If these partial derivatives are continuous at \( z_0 \) and satisfy the Cauchy–Riemann equations at \( z_0 \), then \( f = u + iv \) is differentiable at \( z_0 \).

A complex-valued function that is defined in an open subset \( G \) of the complex plane and is differentiable at every point of \( G \) is said to be holomorphic (analytic, regular) in \( G \).

Our discussion shows that the real and imaginary parts of a holomorphic function have first partial derivatives and satisfy the Cauchy–Riemann equations. Conversely, if the real and imaginary parts of a complex function have continuous first partial derivatives and satisfy the Cauchy–Riemann equations, then the function is holomorphic.

The asymmetry in the two preceding statements - the inclusion of a continuity condition in the second but not in the first relates to an important point. In fact, a holomorphic function is differentiable to all orders, and its real and imaginary parts have continuous partial derivatives to all orders. However we shall only be able to prove this later.

In the meantime, here are some examples to consider.

**Example.** Let \( f(z) = \exp(-1/z^4) \), for \( z \neq 0 \), and let \( f(0) = 0 \). Then \( f \) satisfies the Cauchy–Riemann equations everywhere, but is not continuous (and so not differentiable) at the origin:

\[
\lim_{z=\rho e^{i\theta} \to 0} f(z) = \lim_{x \to 0} e^{-1/x} = \infty.
\]

**Example.** Let \( f(z) = z^5/|z|^4 \), for \( z \neq 0 \), and let \( f(0) = 0 \). Then \( f \) is continuous everywhere and satisfies the Cauchy–Riemann equations at the origin, but fails to be differentiable at the origin.

The following result is an easy to use but hard to prove sufficiency criterion.

**Looman–Menshov theorem** (1923, 1936). Let a function \( f = u + iv \) be defined and continuous in an open set \( G \). If \( u \) and \( v \) have first partial derivatives and satisfy the Cauchy–Riemann equations in \( G \), then \( f \) is holomorphic in \( G \).

**References**


J.D. Gray & S. A. Morris, *When is a function that satisfies the Cauchy–Riemann equations analytic?*