The complex integral

Definition. Let $f(z)$ be a continuous complex function defined on a set containing the trace of a piecewise smooth curve $\gamma(t)$, $a \le t \le b$. The integral of $f(z)$ over γ is

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Writing $f(x+iy) = u(x,y) + iv(x,y)$, $\gamma(t) = x(t) + iy(t)$, $dx = x'(t)dt$, and $dy = y'(t)dt$, we can separate real and imaginary parts:

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b (u+iv)(x'+iy')dt \\ &= \int_a^b (ux' - vy')dt + i\int_a^b (uy' + vx')dt \\ &= \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy).\end{aligned}$$

Linearity in f: $\displaystyle\int_\gamma c_1 f_1(z) + c_2 f_2(z)\ dz = c_1 \int_\gamma f_1(z)dz + c_2 \int_\gamma f_2(z)dz$

Additivity in γ: $\displaystyle\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz,\qquad \gamma_1 = \gamma\big|_{[a,t_1]},\ \gamma_2 = \gamma\big|_{[t_1,b]}$

Examples. The curve $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$, traces the circle $|z - z_0| = r$. We have

$$\int_\gamma 1\ dz = \int_0^{2\pi} \gamma'(t)\ dt = \gamma(2\pi) - \gamma(0) = 0$$

$$\int_\gamma (z - z_0)dz = \int_0^{2\pi} re^{it}\ ire^{it}\ dt = r^2 \int_0^{2\pi} ie^{2it}dt = r^2(e^{4\pi i} - e^{0i})/2 = 0$$

$$\int_\gamma \frac{1}{z - z_0}\ dz = \int_0^{2\pi} \frac{1}{re^{it}}\ ire^{it}\ dt = i\int_0^{2\pi} dt = 2\pi i$$

Choice of parameter. $\displaystyle\int_\gamma f(z)dz$ does not depend on the parameterization of γ.

Proof. Let $\gamma(t)$, $a \le t \le b$, and $\gamma_1(s)$, $a_1 \le s \le b_1$, have the same trace, and let $\gamma_1(s) = \gamma(t(s))$, where $t(s)$ is an increasing differentiable map $[a_1, b_1] \overset{\text{onto}}{\to} [a,b]$. Then

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_{a_1}^{b_1} f(\gamma(t(s)))\gamma'(t(s))t'(s)ds \\ &= \int_{a_1}^{b_1} f(\gamma_1(s))\gamma_1'(s)ds. \qquad \square\end{aligned}$$

Change of direction. $\int_{-\gamma} f(z)dz = -\int_{\gamma} f(z)dz$, where $-\gamma$ is the reverse of γ

Proof. Parameterize the reversed curve $-\gamma$ by $\gamma_-(t) = \gamma(a+b-t)$, $a \leq t \leq b$. Then

$$\begin{aligned}
\int_{-\gamma} f(z)dz &= \int_a^b f(\gamma_-(t))\gamma_-'(t)dt \\
&= \int_a^b f(\gamma(a+b-t))(-1)\gamma'(a+b-t)dt \\
&= \int_b^a f(\gamma(s))(-1)\gamma'(s)(-ds) \\
&= -\int_a^b f(\gamma(s))\gamma'(s)ds \\
&= -\int_\gamma f(z)dz. \qquad \square
\end{aligned}$$

Estimate of the integral. $\left|\int_\gamma f(z)dz\right| \leq \max_{z\in\gamma} |f(z)| \text{ length}(\gamma)$

Proof. Use the triangle inequality for integrals:

$$\begin{aligned}
\left|\int_\gamma f(z)dz\right| &= \left|\int_a^b f(\gamma(t))\ \gamma'(t)\ dt\right| \\
&\leq \int_a^b |f(\gamma(t))\ \gamma'(t)|\ dt \\
&= \int_a^b |f(\gamma(t))|\ |\gamma'(t)|\ dt \\
&\leq M \int_a^b |\gamma'(t)|\ dt \qquad\qquad M = \max_{a\leq t\leq b} |f(\gamma(t))| \\
&= M \text{ length}(\gamma) \qquad \square
\end{aligned}$$

Example. For any real numbers a and b, $\left|\int_{[ia,ib]} e^z dz\right| \leq |a-b|$.

Proof. $\left|\int_{[ia,ib]} e^z dz\right| \leq \max_{t\in[a,b]} |e^{it}| \text{ length}([ia,ib]) = 1 \cdot |a-b| = |a-b|$ $\square$

References.

Stephen Fisher, *Complex variables*.

Donald Sarason, *Complex Function Theory*.