Conformal mapping

Let f be a function holomorphic in an open set G and let $\gamma : [a,b] \to G$ be a curve. Then, at each point t of differentiability of γ,

$$(f \circ \gamma)'(t) = f'(\gamma(t))\,\gamma'(t).$$

To see this, let $\delta \to 0$ in the identity

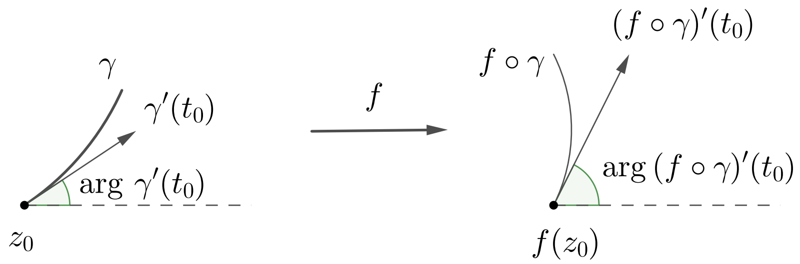
$$\frac{f(\gamma(t+\delta)) - f(\gamma(t))}{\delta} = \frac{f(\gamma(t+\delta)) - f(\gamma(t))}{\gamma(t+\delta) - \gamma(t)}\,\frac{\gamma(t+\delta) - \gamma(t)}{\delta},$$

using the definition of the derivative and the fact that f is differentiable at $\gamma(t)$.

The curve γ is said to be regular at t_0 if $\gamma'(t_0)$ exists and is non-zero, $\gamma'(t_0) \neq 0$. At each point of regularity, γ has a well-defined direction.

Now suppose that $f'(z_0) \neq 0$, $\gamma(t_0) = z_0$, and γ is regular at t_0. Then

$$\arg\,(f \circ \gamma)'(t_0) = \arg f'(z_0) + \arg \gamma'(t_0).$$



This means that $f(z)$ rotates every infinitesimal arc issuing from z_0 by the same angle $\arg f'(z_0)$. Hence the angle between any two regular curves issuing from z_0 is preserved under the mapping. Indeed, let γ_1 and γ_2 be two curves $[0,1] \to G$ issuing from z_0 and regular at $t = 0$. Then the angle between their images at $f(z_0)$ is the same as the angle between the curves at z_0 :

$$\begin{aligned}\arg\,(f \circ \gamma_2)'(0) - \arg\,(f \circ \gamma_1)'(0) &= [\arg f'(z_0) + \arg \gamma_2'(0)] - [\arg f'(z_0) + \arg \gamma_1'(0)] \\ &= \arg \gamma_2'(0) - \arg \gamma_1'(0).\end{aligned}$$

A mapping is said to be conformal if it preserves the size and direction of angles between curves. A holomorphic function is therefore conformal at each point where its derivative does not vanish. The conjugation map $z \mapsto \overline{z}$ reverses the angles, it is anti-conformal.

A conformal map is locally one-to-one. The function $f(z) = z^2$ is conformal at each point other than the origin, it is univalent locally but not globally.

It can be shown that if a complex-valued function f is conformal in an open region and is differentiable in the real sense (which is the case when its first partial derivatives are continuous), then f is holomorphic in the region and its derivative is never zero.

Continuing in the same vein, we can also give a geometric interpretation to $|f'(z_0)| \neq 0$. The equality

$$|(f \circ \gamma)'(t_0)| = |f'(z_0)|\,|\gamma'(t_0)|$$

shows that every infinitesimal arc issuing from z_0 is scaled under f by the same factor of $|f'(z_0)|$, independent of the direction. Thus, taking into account conformality, we see that, locally at z_0, the mapping $f(z)$ is very nearly a similarity transformation with a coefficient of distortion given by $|f'(z_0)|$.

Note that the function $f(z) = z^2$ is not conformal and not a similarity at the origin. In fact, it doubles the angles and squares the magnitudes at $z_0 = 0$. The assumption $f'(z_0) \neq 0$ is therefore necessary.

Not also that a holomorphic mapping $f(x + iy) = u(x, y) + iv(x, y)$, viewed as a vector function of two real variables $(x, y) \mapsto (u, v)$, has the Jacobi matrix

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}.$$

The linear transformation performed by this matrix (the linear part of the mapping) is a composition of a rotation by an angle of $\arg f'(x + iy)$ and a scaling by a factor of $|f'(x + iy)|$ at every point where the determinant $|f'|^2 = u_x^2 + v_x^2$ is nonzero.

References

George Polya & Gordon Latta, *Complex Variables.*

Donald Sarason, *Complex Function Theory.*