Conformal vs Holomorphic

Tangent vector to the image curve

Let f be a holomorphic function defined in an open set G and let γ be a differentiable curve contained in G. Fix a point $z_0 = \gamma(t_0)$. Then the tangent vector to the image of γ under f at the point $f(z_0)$ is given by

$$(f \circ \gamma)'(t_0) = f'(z_0)\gamma'(t_0).$$

Proof. As $t \to t_0$, $\gamma(t) \to z_0$, by continuity. In turn, $\dfrac{f(\gamma(t)) - f(z_0)}{\gamma(t) - z_0} \to f'(z_0)$, by complex differentiability. Hence

$$(f \circ \gamma)'(t_0) = \lim_{\Delta t \to 0} \frac{\Delta f(\gamma(t))}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta f(\gamma(t))}{\Delta \gamma(t)} \frac{\Delta \gamma(t)}{\Delta t} = f'(z_0)\gamma'(t_0). \qquad \square$$

In particular, if $f'(z_0) \neq 0$ and $\gamma'(t_0) \neq 0$, we see that f rotates γ at z_0 by an angle of $\arg f'(z_0)$. So a holomorphic function is conformal at each point where its derivative is nonzero. Indeed, if γ_1 and γ_2 are regular curves in G and $z_0 = \gamma_1(t_1) = \gamma_2(t_2)$, then

$$\begin{aligned} \arg(f \circ \gamma_2)'(t_2) - \arg(f \circ \gamma_1)'(t_1) &= [\arg f'(z_0) + \arg \gamma_2'(t_2)] - [\arg f'(z_0) + \arg \gamma_1'(t_1)] \\ &= \arg \gamma_2'(t_2) - \arg \gamma_1'(t_1), \end{aligned}$$

provided $f'(z_0) \neq 0$.

Linear functions of z

Consider a function of the form $f(z) = cz$, where c is a nonzero complex number. Such a function is holomorphic in all of $\mathbb{C}$. The mapping $z \mapsto cz$ is the composition of two interchangeable operations: scaling by a factor of $|c|$ and rotation by an angle of $\arg c$ about the origin,

$$z \mapsto |c|z \mapsto cz,$$

Note that f maps radial rays onto radial rays and preserves the angles between them.

Linear functions of x and y

Consider a function of the form $f(z) = ax + by$, where a, b is are complex numbers. An equivalent expression is

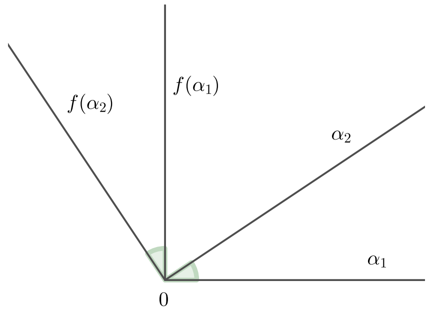
$$f(z) = \frac{a - ib}{2}\, z + \frac{a + ib}{2}\, \overline{z} = c_1 z + c_2 \overline{z}.$$

Such a function is holomorphic if and only if $c_2 = 0$. The mapping $x + iy \mapsto ax + by$ maps radial rays onto radial rays, provided a and b are not on the same line through the origin. Indeed, the image of the ray $\arg z = \theta$ is the ray $\arg z = \arg(a \cos\theta + b \sin\theta)$.

Lemma

If $f(z) = c_1 z + c_2 \overline{z}$ is conformal at the origin, then $c_1 \neq 0$ and $c_2 = 0$.

Proof. For any two radial rays α_1 and α_2 forming a given angle, the angle between their images $f(\alpha_1)$ and $f(\alpha_2)$ is the same. Therefore f is interchangeable with rotations about the origin.

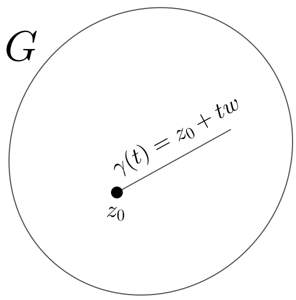


In particular, f is interchangeable with the $90°$ rotation, $f(iz) = if(z)$. But the identity $c_1 iz + c_2 \overline{iz} = i(c_1 z + c_2 \overline{z})$ yields $-c_2 = c_2$. Hence $c_2 = 0$ and $c_1 \neq 0$. $\square$

Theorem

Let f be defined in an open set G and be differentiable in the real sense.
If f is conformal, then it is holomorphic in G and its derivative is nonzero.

Proof. Fix $z_0 \in G$. For a fixed nonzero complex number w, let $\gamma(t) = z_0 + tw$, $0 \leq t \leq t_1$, be a straight line segment in G issuing from $z_0 = \gamma(0)$.



With these provisions, $\Delta t = t$, $\Delta\gamma(t) = tw$, and $\Delta f(\gamma(t)) = f(\gamma(t)) - f(z_0)$.
Note that $o(\Delta\gamma(t)) = o(t)$, as $t \to 0$. By real differentiability,

$$\frac{\Delta f(\gamma(t))}{\Delta t} = \frac{c_1 \Delta\gamma(t) + c_2 \Delta\overline{\gamma(t)} + o(\Delta\gamma(t))}{t} \to c_1 w + c_2 \overline{w}, \quad t \to 0.$$

So the tangent vector to the curve $f \circ \gamma$ at the point $f(z_0)$ is given by $c_1 w + c_2 \overline{w}$.
Since f is conformal at z_0, the map $w \mapsto c_1 w + c_2 \overline{w}$ is conformal at the origin.
Therefore $c_2 = 0$ (Cauchy–Riemann equations hold at z_0) and $c_1 = f'(z_0) \neq 0$. $\square$