A NOTE ON CHEBYSHEV POLYNOMIALS AND FINITE DIFFERENCE WAVE EQUATION

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Abstract. We discuss a positivity property of the Chebyshev polynomials of the second kind and its connection with the finite difference wave equation.

1. Introduction

Let Q be an n-th degree polynomial and let Q_m be the coefficients of the Fourier cosine expansion

$$Q(1-s+s\cos\theta) = Q_0(s) + 2\sum_{m=1}^{n} Q_m(s)\cos m\theta.$$

Furthermore, let the coefficients Q_m satisfy the following positivity property:

$$Q_m(s) \geq 0, \qquad m = 0, \ldots, n, \quad 0 < s < 1. \tag{1}$$

What can be then said of the polynomial Q?

The property (1) is enjoyed by the Legendre polynomials P_n, by the Chebyshev polynomials of the second kind U_n, and, more generally, by all ultraspherical polynomials C_n^ν, where ν is a positive integer. In fact, every linear combination of the Legendre polynomials with nonnegative coefficients necessarily satisfies (1) [3].

In this note we focus on $Q = U_n$ as polynomials with property (1) in view of the fact that, for each $0 < s < 1$, the cosine coefficients $Q_m = U_{m,n}(s)$ of $U_n(1-s+s\cos\theta)$ satisfy a finite difference wave equation. Thus, not only (1) receives a vivid interpretation in this case, but one also obtains a nontrivial property of a wave propagating on a discrete lattice.

The positivity of $U_{m,n}$ ensures the monotonic behavior of the de Branges weight functions [1] and, as such, has been a theme of many publications. At the same time, the wave equation aspect of this positivity seems to have been left out of consideration.

The presentation below is elementary and includes some known and classical facts for reader's convenience.

2. Chebyshev polynomials of the second kind

The Chebyshev polynomials of the second kind U_n are defined by the expansion

$$\frac{1}{1-2\cos\theta\ z+z^2} = \sum_{n=0}^{\infty} U_n(\cos\theta) z^n, \tag{2}$$

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valid for $|z| < 1$. Letting $\lambda = e^{i\theta}$, one easily finds the explicit representation

$$U_n = \lambda^n + \lambda^{-n} + \lambda^{n-2} + \lambda^{2-n} + \cdots = \frac{\lambda^{n+1} - \lambda^{-n-1}}{\lambda - \lambda^{-1}}$$

and the recurrence relation

$$(\lambda + \lambda^{-1})U_n = U_{n+1} + U_{n-1}, \quad n = 1, 2, 3, \ldots. \tag{3}$$

For a parameter $0 < s < 1$, replace $\cos\theta$ by the convex linear combination $(1-s) + s\cos\theta$, or, equivalently, consider the change of variable

$$\lambda + \lambda^{-1} \mapsto 2(1-s) + s(\lambda + \lambda^{-1}).$$

Then every U_n can be expanded by the powers of λ,

$$\begin{aligned} U_n &= U_{0,n} + \sum_{m=1}^{n} U_{m,n}(\lambda^m + \lambda^{-m}) \\ &= U_{0,n} + 2\sum_{m=1}^{n} U_{m,n}\cos m\theta. \end{aligned} \tag{4}$$

The first three of these expansions are

$$\begin{aligned} &U_0 = 1, \\ &U_1 = 2(1-s) + s(\lambda + \lambda^{-1}), \\ &U_2 = 4(1-s)^2 + 2s^2 - 1 + 4(1-s)s(\lambda + \lambda^{-1}) + s^2(\lambda^2 + \lambda^{-2}). \end{aligned}$$

Relation (3) yields the following recurrences for the coefficients $U_{m,n}$:

$$\begin{cases} U_{0,n+1} + U_{0,n-1} = 2(1-s)\ U_{0,n} + 2s\ U_{1,n}, \\ U_{m,n+1} + U_{m,n-1} = 2(1-s)\ U_{m,n} + s\ (U_{m-1,n} + U_{m+1,n}),\ 1 \le m < n, \\ U_{n,n+1} = 2(1-s)\ U_{n,n} + s\ U_{n-1,n}, \\ U_{n+1,n+1} = s\ U_{n,n}. \end{cases} \tag{5}$$

It is immediate from the last two relations in (5) that

$$U_{n,n} = s^n \quad \text{and} \quad U_{n-1,n} = 2n(1-s)s^{n-1}.$$

In fact, using the generating function (2) it is not hard to see that

$$U_{m,n} = \sum_{k=m}^{n} (-1)^{k-m} s^k \binom{n+k+1}{n-k}\binom{2k}{k-m}, \quad m = 0, \ldots, n.$$

One has the following nonobvious property of these combinatorial sums: the coefficients $U_{m,n}$ defined by (4) are nonnegative. The cause of the positivity and its significance are explained next.

3. The positivity of $U_{m,n}$

The celebrated de Branges theorem [1] relies on the fact that certain weight functions are monotone decreasing on $[0, \infty)$. In the case needed for the Bieberbach conjecture, these weights can be expressed (for every $n \ge 1$) as

$$\begin{aligned} \sigma_m(t) &= \frac{1}{2\pi}\int_t^\infty \int_0^{2\pi} U_n(1 - e^{-\tau} + e^{-\tau}\cos\theta)\cos m\theta\ d\theta d\tau \\ &= \int_t^\infty U_{m,n}(e^{-\tau})d\tau, \quad m = 1, \ldots, n, \end{aligned}$$

so that their monotonicity is a matter of U_n satisfying (1) (see [3]). The most general de Branges weight functions can be obtained by replacing U_n by a polynomial Q satisfying (1), and an elementary description of all such Q is given by the following result.

Theorem 1 ([3]). *A polynomial Q satisfies (1) if and only if the coefficients a_k in the Legendre polynomial expansion $Q(x) = \sum_{k=0}^{n} a_k P_k(x)$ satisfy*

$$\min_{0\le x\le 1} \sum_{k=m}^{n} a_k \, \frac{(k-m)!}{(k+m)!} \, P_k^{(m)}(x)^2 \ge 0, \quad m = 0, \dots, n.$$

In particular, all positive linear combinations of the Legendre polynomials satisfy (1).

The reason behind Theorem 1 is the classical addition theorem for the Legendre polynomials, which is a manifestation of the rotation invariance of the distance in three dimensions [5]. The Legendre polynomials $P_n(x)$ are defined by the expansion

$$\frac{1}{\sqrt{1-2xz+z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n, \tag{6}$$

valid for small z, and the addition theorem states that

$$\begin{aligned} &P_n(\cos a \cos a_1 + \sin a \sin a_1 \cos\theta) \\ &\quad = P_n(\cos a) P_n(\cos a_1) + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_{m,n}(\cos a) P_{m,n}(\cos a_1) \cos(m\theta), \end{aligned} \tag{7}$$

where $P_{m,n}(x) = (1-x^2)^{m/2} P_n^{(m)}(x)$ are the associated Legendre functions. The case $a = a_1$ of (7) implies that P_n satisfies (1), and hence the same is true of all ultraspherical (Gegenbauer) polynomials C_n^{ν} generated by the positive integer powers ν of the generating function (6). In particular, the Chebyshev polynomials $C_n^2 = U_n$ are in.

Remark 1. The idea to use (7) to treat $\sigma_m(t)$ was utilized in [6]. In [1] the Askey-Gasper inequalities were used for the same purpose. The addition theorem for U_n (together with the classical fact that $P_m(\cos\theta)$ has nonnegative cosine coefficients) allows one to establish the positivity of $U_{m,n}$ directly. The latter approaches however are more complex than (7). They also lack a natural three-dimensional illustration.

4. The finite difference wave equation

The finite-difference analog of the one-dimensional wave equation $u_{xx} - u_{yy} = 0$ [2] is given by

$$\frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{h^2} - \frac{u_{m,n+1} - 2u_{m,n} + u_{m,n-1}}{\delta^2} = 0, \tag{8}$$

where $u_{m,n} = u(mh, n\delta)$ is a discrete function defined on the lattice of mesh points $(x, y) = (mh, n\delta)$, m, n are integers, and $h, \delta > 0$ are the mesh sizes of the grid.

Rearranging the terms in (8) one finds that

$$u_{m,n+1} + u_{m,n-1} = 2(1 - \delta^2/h^2) u_{m,n} + \delta^2/h^2 (u_{m+1,n} + u_{m-1,n}).$$

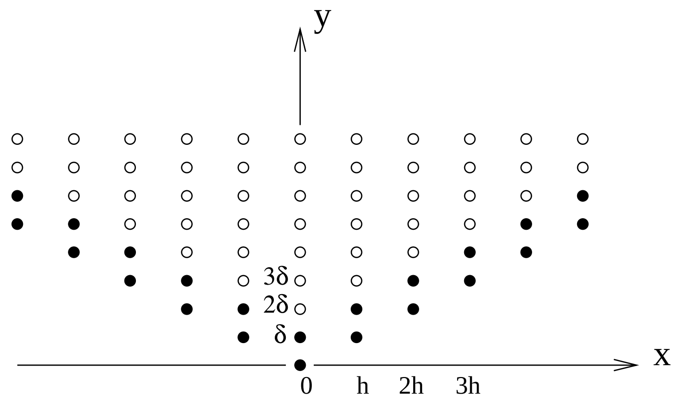

FIGURE 1. The grid for (8) & (9).

These are the relations (5) for the coefficients $U_{m,n}$, with $s = \delta^2/h^2$. To get an exact match one should choose $\delta \le h$ (the von Neumann stability criterion), the initial conditions

$$\begin{cases} u_{n,n} = s^n, \quad n \ge 0, \\ u_{n-1,n} = 2n(1-s)s^{n-1}, \quad n \ge 1, \end{cases} \tag{9}$$

and allow the symmetric reflection across the y-axis by letting $U_{-m,n} = U_{m,n}$ (see Figure 1).

The problem (8) & (9) is well-posed as the values $u_{m,n}$ can be obtained inductively by proceeding upward from row to row. On the other hand, it is not a discrete analog of the Cauchy problem: the boundary data is given along the characteristics, and the values

$$u(x,0), \quad u_y(x,0)$$

are not readily available. The behavior of the solution is oscillatory, as one would expect from a wave, and its positivity is not an obvious matter. Nevertheless, we have the following theorem, equivalent to the positivity of $U_{m,n}$.

Theorem 2. *The solution of the finite difference problem* (8) *&* (9) *is nonnegative at the lattice points* $(mh, n\delta)$ *in the region* $\delta|x| \ge hy \ge 0$.

We conclude by two reasonably natural questions. First, it may be of interest to find an elementary proof of Theorem 2 based solely on the properties of the wave equation (if such a proof exists). See [4] in this connection. Second, it may also be of interest to characterize all sets of the initial conditions $u_{n,n}, u_{n-1,n}$ that yield a nonnegative solution of (8).

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