Examples of mappings

Example 1. The function $f(z) = |z|$ is defined for all $z \in \mathbb{C}$ and takes values on the non-negative real axis:

$$\text{domain } f = \mathbb{C}, \qquad \text{range } f = [0, +\infty).$$

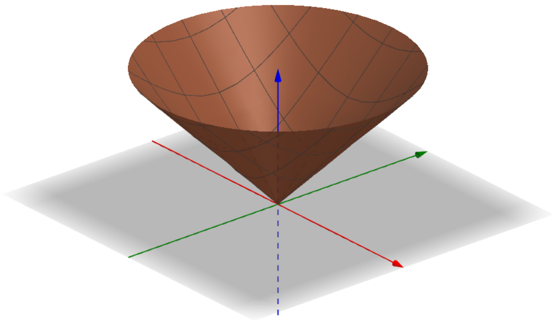
Thus the entire plane collapses into a ray under $f(z)$,

$$\operatorname{Re} f(z) \geq 0, \qquad \operatorname{Im} f(z) = 0.$$

Along each circle $|z| = r$ centered at the origin, $f(z)$ is constant. The graph of $f(z)$ can be visualized in three dimensions. The surface

$$(x, y, \sqrt{x^2 + y^2})$$

is a circular cone.



Example 2. The function $f(z) = \operatorname{Arg} z$ is defined for all nonzero $z \in \mathbb{C}$ and takes values on an interval:

$$\text{domain } f = \mathbb{C} \setminus \{0\}, \qquad \text{range } f = (-\pi, \pi].$$

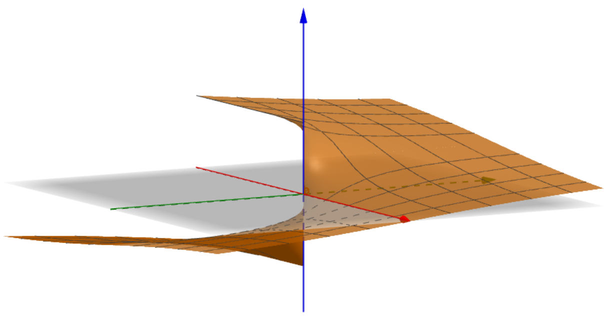
Along each ray $\arg z = \theta_0$, $-\pi < \theta_0 \leq \pi$, issuing from the origin, $f(z)$ is constant:

$$\operatorname{Re} f(z) = \theta_0, \qquad \operatorname{Im} f(z) = 0.$$

The graph of $f(z)$ can also be visualized in three dimensions. The surface

$$(x, y, \operatorname{Arg}(x + iy))$$

is reminiscent of a spiral staircase and suffers a break along the non-positive real axis.



Example 3. The function $f(z) = z^2$ is defined for all $z \in \mathbb{C}$. Since for each $w \in \mathbb{C}$, the equation $z^2 = w$ has a solution, the range of f is all of $\mathbb{C}$ as well. The correspondence

$$(x, y) \mapsto (x^2 - y^2, 2xy)$$

cannot be visualized in three dimensions, so we have to describe the mapping by other means. Writing $z = r(\cos\theta + i\sin\theta)$, we have

$$f(z) = r^2(\cos 2\theta + i\sin 2\theta).$$

Therefore the mapping $z \mapsto z^2$ has the following properties:

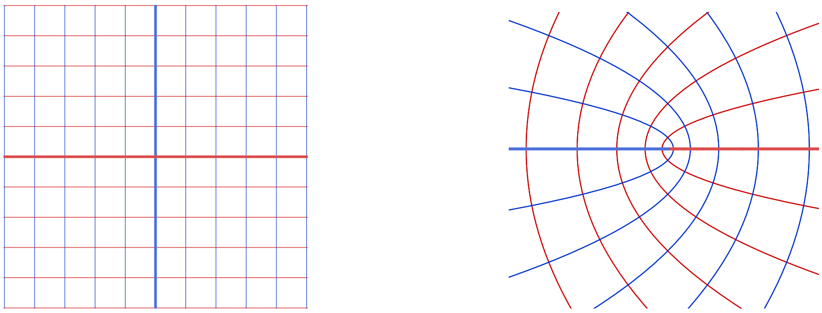
- the argument of each nonzero z doubles
- the points with $0 < |z| < 1$ move closer to the origin
- the points with $|z| > 1$ move farther from the origin
- the unit circle $|z| = 1$ remains invariant under the map

Writing

$$\operatorname{Re} f(z) = x^2 - y^2, \quad \operatorname{Im} f(z) = 2xy,$$

we see that the coordinate grid lines $x = x_0$ and $y = y_0$ are bent into parabolas. More precisely:

- the real axis $(x, 0)$, $-\infty < x < \infty$, folds into the non-negative real axis $(x^2, 0)$
- for $y_0 \neq 0$, the horizontal line (x, y_0) bends into the parabola $(x^2 - y_0^2, 2xy_0)$
- the imaginary axis $(0, y)$, $-\infty < y < \infty$, folds into the non-positive real axis $(-y^2, 0)$
- for $x_0 \neq 0$, the vertical line (x_0, y) bends into the parabola $(x_0^2 - y^2, 2x_0 y)$



We can also determine which curves are mapped to the coordinate grid lines. Setting the real/imaginary part of $f(z) = u + iv$ to a constant value,

$$\operatorname{Re} f(z) = u_0, \quad \operatorname{Im} f(z) = v_0.$$

we see that the preimages of the coordinate grid lines are hyperbolic arcs. More precisely:

- the preimage of the real axis $(u, 0)$, $-\infty < u < \infty$, is the coordinate cross $xy = 0$
- for $v_0 \neq 0$, preimage of the horizontal line (u, v_0) is the hyperbola $xy = v_0/2$
- the preimage of the imaginary axis $(0, v)$, $-\infty < v < \infty$, is the orthogonal cross $y = \pm x$
- for $u_0 \neq 0$, the preimage of the vertical line (u_0, v) is the hyperbola $x^2 - y^2 = u_0$