A note on exponential limits
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**LIMITS OF TYPE \((1 + 0)^{\infty}\)**

If \(n\) is an integer, or any real variable approaching positive or negative infinity, 

\[
\lim_{n \to \pm \infty} \left(1 + \frac{1}{n}\right)^n = e. \tag{*}
\]

To verify this, consider the logarithm 

\[g(n) = \ln \left(1 + \frac{1}{n}\right)^n = n \ln \left(1 + \frac{1}{n}\right).\]

and make a change of variable, \(h = \frac{1}{n}\). Then 

\[
\lim_{n \to \pm \infty} g(n) = \lim_{h \to 0} \frac{\ln(1 + h)}{h} = 1
\]

is simply the slope of \(y = \ln x\) at \(x = 1\).

Consequently \(e^{g(n)} = \left(1 + \frac{1}{n}\right)^n\) approaches \(e^1 = e\) as \(n \to \infty\).

Formula \((*)\) can be written as 

\[
\lim_{\Box \to \pm \infty} (1 + 1/\Box)^\Box = e.
\]

It holds true for any expression \(\Box\) tending to \pm infinity and can be used to evaluate exponential limits.

**EXAMPLE 1.** Evaluate \(\lim_{n \to \infty} (1 + \frac{2}{n})^{-3n}\).

**Solution.** If \(x = \frac{n}{2}\) then \(x \to \infty\) and 

\[
\lim_{n \to \infty} (1 + \frac{2}{n})^{-3n} = \lim_{x \to \infty} (\left(1 + \frac{1}{x}\right)^x)^{-3} = e^{-6}.
\]

**EXAMPLE 2.** Evaluate \(\lim_{n \to \infty} (1 - \frac{1}{n})^n\).

**Solution.** Write \(m = -n\):

\[
\lim_{n \to \infty} (1 - \frac{1}{n})^n = \lim_{m \to -\infty} (\left(1 + \frac{1}{m}\right)^m)^{-1} = e^{-1}.
\]

**EXPOENTIAL versus POWER GROWTH**

Another limit technique is illustrated by the following example.

**EXAMPLE 3.** Evaluate \(\lim_{n \to \infty} \frac{n^2}{2^n}\).

**Solution.** Let \(a_n = \frac{n^2}{2^n}\) and consider successive ratios 

\[
\frac{a_{n+1}}{a_n} = \frac{(n + 1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2.
\]

As \(n \to \infty\), the ratios \(\frac{a_{n+1}}{a_n}\) are monotonically decreasing to \(\frac{1}{2}\).

In particular, for all indices \(n\) starting with some \(N\), 

\[
\frac{a_{n+1}}{a_n} < \frac{3}{4}.
\]
It follows that
\[ a_{N+1} < \frac{3}{4} a_N \]
\[ a_{N+2} < \frac{3}{4} a_{N+1} < \left( \frac{3}{4} \right)^2 a_N \]
\[ \ldots \]
\[ a_{N+k} < \frac{3}{4} a_{N+k-1} < \ldots < \left( \frac{3}{4} \right)^k a_N \]
\[ \ldots \]
Thus, for all \( k = 1, 2, 3, \ldots \), terms \( a_{N+k} \) are dominated by \( \left( \frac{3}{4} \right)^k a_N \).

As \( k \to \infty \), \( \left( \frac{3}{4} \right)^k a_N \) decreases to 0 (exponentially fast).

Terms of the original sequence therefore converge to 0 as well.

The last example makes use of l'Hôpital’s rule.

**EXAMPLE 4.** Evaluate \( \lim_{n \to \infty} (1 + n)^{1/n} \).

**Solution.** This limit is of type \( \infty^0 \). Observe that, by l'Hôpital’s rule,
\[ \lim_{n \to \infty} \ln(1 + n)^{1/n} = \lim_{n \to \infty} \frac{\ln(1 + n)}{n} = \lim_{n \to \infty} \frac{1}{1+n} = 0. \]

Therefore
\[ \lim_{n \to \infty} (1 + n)^{1/n} = e^0 = 1. \]