Complex Analysis

Grinshpan

The geometric series and its derivative

The power series $\sum_{n=0}^{\infty} z^n$ converges for $|z| < 1$ and diverges for $|z| \geq 1$.

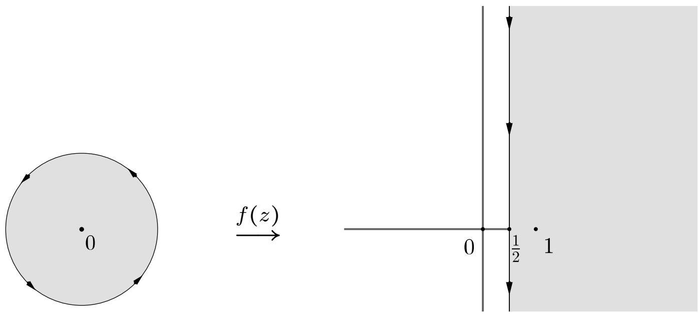
For $|z| < 1$, it represents the holomorphic function $f(z) = \dfrac{1}{1-z}$.

On each subdisk $|z| \leq \rho$, $0 < \rho < 1$, the convergence is absolute and uniform:

$$\left|\sum_{n=0}^{\infty} z^n\right| \leq \sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho} \qquad \text{and} \qquad \left|\sum_{k=0}^{n} z^k - \frac{1}{1-z}\right| \leq \frac{\rho^{n+1}}{1-\rho} \to 0, \quad n \to \infty.$$

For $z = e^{i\theta}$, $\quad f(z) = \dfrac{e^{-i\frac{\theta}{2}}}{-2i\sin\frac{\theta}{2}} = \dfrac{1}{2} + \dfrac{i}{2}\cot\dfrac{\theta}{2}.$

The linear-fractional map $f(z)$ sends the unit disk onto the half-plane $\Re z > 1/2$.



The term-by-term derivative $\sum_{n=1}^{\infty} nz^{n-1}$ converges for $|z| < 1$ and diverges otherwise.

For $|z| < 1$, it represents the holomorphic function $f'(z) = \dfrac{1}{(1-z)^2} = f^2(z)$.

For $z = e^{i\theta}$, $\quad f'(z) = \dfrac{1}{4} - \dfrac{1}{4}\cot^2\dfrac{\theta}{2} + \dfrac{i}{2}\cot\dfrac{\theta}{2}.$

So $f'(z)$ maps $|z| < 1$ univalently and conformally onto the region $\Re z > 1/4 - (\Im z)^2$.