# Green's theorem for a coordinate rectangle

Green's theorem relates the line and area integrals in the plane.

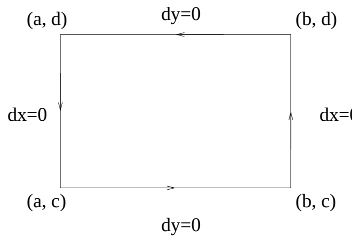
Let $C$ be a piecewise smooth, simple closed curve and let $D$ be the open region enclosed by $C$. Let $P(x,y)$ and $Q(x,y)$ be continuously differentiable functions in an open set containing $D$. Then

$$\int_C P dx + Q dy = \iint_D (Q_x - P_y) dx dy,$$

where $C$ is given the counterclockwise orientation.
Equivalently, if we write $\overrightarrow{F} = (P,Q)$ and $d\overrightarrow{r} = (dx,dy)$, then

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = \iint_D \text{curl}\, \overrightarrow{F} dx dy.$$

Green's formula is transparent for a rectangle with sides parallel to the coordinate axes. Indeed, let $D = [a,b] \times [c,d]$.




Then

$$\begin{aligned}
\iint_D Q_x(x,y) dx dy &= \int_c^d \int_a^b Q_x(x,y) dx dy = \int_c^d (Q(b,y) - Q(a,y)) dy \\
&= \int_{(b,c)}^{(b,d)} Q(x,y) dy + \int_{(a,d)}^{(a,c)} Q(x,y) dy \\
&= \int_C Q(x,y) dy
\end{aligned}$$

and

$$\begin{aligned}
\iint_D P_y(x,y) dx dy &= \int_a^b \int_c^d P_y(x,y) dy dx = \int_a^b (P(x,d) - P(x,c)) dx \\
&= -\int_{(b,d)}^{(a,d)} P(x,y) dx - \int_{(a,c)}^{(b,c)} P(x,y) dx \\
&= -\int_C P(x,y) dx.
\end{aligned}$$

Hence, combining the two calculations together, we obtain

$$\iint_D (Q_x - P_y) dx dy = \int_C P dx + Q dy.$$

The general case may be obtained by partitioning $D$ into small squares (some will have curvilinear sides) and adding up the circulations around each square.