Differential Equations
Grinshpan

**Homogeneous Equations.**

A function $f(x, y)$ is said to be homogeneous of degree 0 if

$$f(tx, ty) = f(x, y)$$

for all real $t$. Such a function only depends on the ratio $y/x$ (because $f(x, y) = f(1/x, 1/x) = f(1, y/x)$) and one can write $f(x, y) = h(y/x)$.

A first order differential equation

$$y'(x) = f(x, y)$$

is called homogeneous if $f(x, y)$ is homogeneous of degree 0.

**Caution.** In the context of linear equations the term *homogeneous* has a completely different meaning.

The equation

$$y'(x) = h(y/x),$$

if not already separable, can always be transformed into a separable equation. Indeed, let $v = y/x$. Then $y' = (xy)' = v + xv'$, and we obtain

$$v + xv' = h(v),$$

which evidently is separable:

$$\frac{dv}{h(v) - v} = \frac{dx}{x}.$$

If it is possible to find $v$ by integration, one can recover $y = xv$.

**Example.** The equation

$$y'(x) = \frac{x^2 + y^2}{xy}$$

can be written as $y'(x) = \frac{x}{y} + \frac{y}{x} = \frac{1}{y/x} + \frac{y}{x}$ and so is homogeneous. Letting $v = y/x$, we obtain

$$v + xv' = 1/v + v$$

$$xv' = 1/v$$

$$vdv = \frac{dx}{x}$$

$$v^2/2 = \ln |x| + k$$

$$v^2 = \ln(x^2) + 2k.$$ 

Hence

$$y^2 = x^2(\ln(x^2) + c).$$

For a fixed $c$, the last equation implicitly defines four solutions:

$$y(x) = \pm x \sqrt{\ln(x^2) + c}, \quad x > e^{-c/2},$$

$$y(x) = \pm x \sqrt{\ln(x^2) + c}, \quad x < -e^{-c/2}.$$