Homogeneous Equations

A function \( f(x, y) \) is said to be homogeneous of degree 0 if
\[
f(tx, ty) = f(x, y)
\]
for all real \( t \). Such a function only depends on the ratio \( y/x \):
\[
f(x, y) = f(x/x, y/x) = f(1, y/x)
\]
and we can write \( f(x, y) = h(y/x) \).

A first-order differential equation \( y'(x) = f(x, y) \) is called homogeneous if \( f(x, y) \) is homogeneous of degree 0.

Caution In the context of linear equations the term homogeneous has a different meaning.

The equation
\[
y'(x) = h(y/x),
\]
if not already separable, can always be transformed into a separable equation. Indeed, let \( v = y/x \). Then \( y' = (xv)' = v + xv' \), and our differential equation becomes
\[
v + xv' = h(v),
\]
which evidently is separable:
\[
\frac{dv}{h(v) - v} = \frac{dx}{x}.
\]
If it is possible to find \( v \) by integration, we can recover \( y = xv \).

Example The equation \( y'(x) = \frac{x^2 + y^2}{xy} \) is homogeneous; it can be written as \( y' = x/y + y/x \). Letting \( v = y/x \), we have:
\[
v + xv' = 1/v + v
\]
\[
sv' = 1/v
\]
\[
vdv = dx/x
\]
\[
v^2/2 = \ln |x| + c
\]
\[
v^2 = \ln(x^2) + 2c.
\]
Hence
\[
y^2 = x^2 (\ln(x^2) + \text{const}) .
\]
The obtained formula implicitly defines several solution curves.