A Chaotic Search for i
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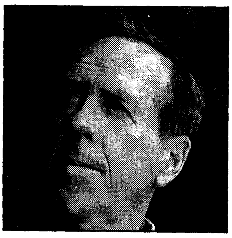
A Chaotic Search for i

Gilbert Strang



Gilbert Strang's teaching at M. I. T. has led to three textbooks —on linear algebra, on applied mathematics, and this year (1991) on calculus. The calculus book includes an optional section on iterations, which are quick to compute (and also important). Normally they illustrate convergence. In this instance they illustrate chaos.

The study of iterations is not a new fad. It is central to the applications of mathematics, whether we are computing π or the roots of a cubic or the eigenvalues of a matrix. In most cases the iterations converge as hoped. Sometimes they diverge (Newton's method provides beautiful and instructive examples). A third possibility is oscillation between two or more limits—convergence to a "cycle." What is new in this theory is a fourth possibility—divergence but not divergence to infinity. The sequence $\{x_n\}$ can oscillate in a random and unpredictable way, even though it is determined by a clear and simple rule $x_{n+1} = F(x_n)$.

This note is about one specific example—a perfectly natural iteration with unexpected results. The output looks random, but we can find a formula. I shouldn't have called it unpredictable. The source of the example is again **Newton's method**, applied to the equation $x^2 - b = 0$. The iteration for $\sqrt{b}$ was known to the Babylonians (at least to a few Babylonians):

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{b}{x_n}\right). \tag{1}$$

The chaotic example appears when we go outside the rules, and choose $b = -1$. Newton's method is being asked to do the impossible—to solve $x^2 + 1 = 0$. The real numbers x_n cannot approach the imaginary number i (or $-i$). The method will fail, and the question is how.

To solve $f(x) = 0$, Newton follows the tangent lines. Quadratic convergence is normal. The underlying idea is the most basic approximation of calculus—which has three useful forms:

$$\frac{df}{dx} \approx \frac{\Delta f}{\Delta x} \qquad \Delta f \approx \frac{df}{dx}\Delta x \qquad \Delta x \approx \frac{\Delta f}{df/dx}.$$

The first leads to the definition of df/dx. The second is linear approximation, when the derivative is known. In the third we know the slope and the desired Δf. To go from $f(x_n)$ to zero we want $\Delta f = -f(x_n)$. Then Newton (or Raphson) chose

$$\Delta x = \Delta f/\text{slope} \quad \text{or} \quad x_{n+1} - x_n = -f(x_n)/f'(x_n). \tag{2}$$

Specialized to $f(x) = x^2 - b$ this is (1). The choice $b = -1$ is the subject of this paper:

$$x_{n+1} = F(x_n) = \frac{1}{2}\left(x_n - \frac{1}{x_n}\right). \tag{3}$$

Figure 1 shows the good case $b > 0$. Figure 2 shows $b = -1$, with two special starting points.

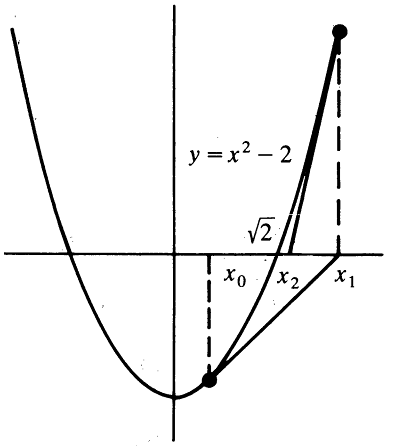


Figure 1
Newton converges quadratically.

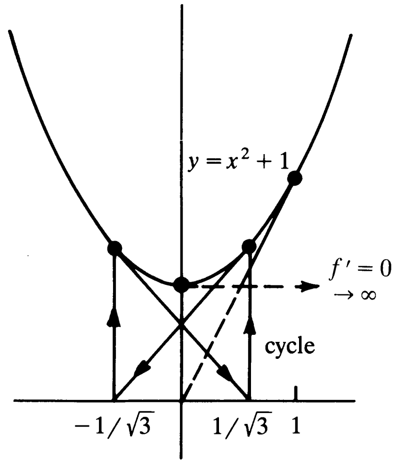


Figure 2
Trouble at $f' = 0$,
cycle at $x = 1/\sqrt{3}$.

Example 1. Choose $x_0 = 1$. Then $x_1 = 0$. Then x_2 is infinite (division by zero). At first I thought most starting values x_0 would lead toward infinity.

Example 2. Choose $x_0 = 1/\sqrt{3}$. Then $x_1 = -1/\sqrt{3}$. Then $x_2 = 1/\sqrt{3}$. The cycle has "period 2," which means that $x_{n+2} = x_n$.

Figure 3 shows this cycle numerically (keeping 16 decimals). The oscillation between $1/\sqrt{3}$ and $-1/\sqrt{3}$ looks fine to the naked eye. But a small roundoff error is growing—which destroys the cycle.

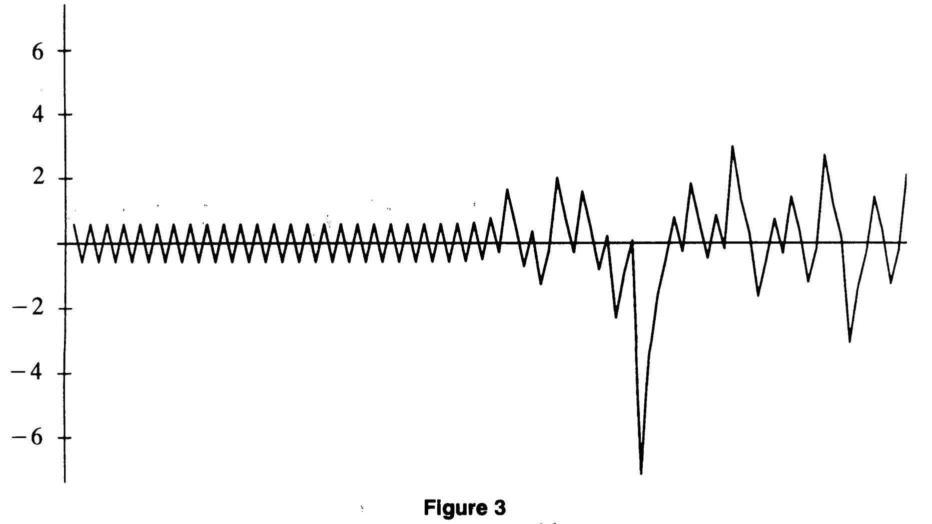


Figure 3
The cycle is unstable.

Example 3. Choose $x_0 = 1000$. Then x_1 is close to 500. Then x_2 is close to 250. When the sequence gets near zero, the larger term $1/x_n$ throws it far out again. The x_n will not diverge to infinity, because any large x_n is approximately cut in half at the following step.

Example 3 begins to show numerically what we generally see graphically—confusion. I started down this path after writing a straightforward description of Newton's method [**4**]. Any calculus textbook must emphasize the practical importance of this algorithm: *When we teach Newton's method we are teaching the real thing*. It is the method of choice, in a tremendous variety of applications. The standard examples like $x^2 - 2 = 0$ show the quadratic convergence that makes the method so powerful. The error drops from 10^{-2} to 10^{-4} to 10^{-8} in two steps—and not only in this simple one-dimensional example. But after a string of successes, $x^2 + 1 = 0$ looked like a total failure.

Some part of every mathematics course should be ongoing and incomplete and alive—and preferably also attractive. Most of calculus was settled long ago. It is beautiful, and it gives us more than enough to do. But students naturally think that formulas are everything—our subject seems enclosed in parentheses. They can't imagine that the discovery process continues, unless they have an example. We need an idea that is easily explained (but not easily exhausted). It is a good thing if the idea can be tried on a calculator or computer—because that way it becomes real. I recommend this iteration.

Note. The intended example was the iteration $z_{n+1} = az_n - az_n^2$. This is the "humble parabola" that leads to chaos. It is easy to execute, and it is full of good mathematics. By varying the number a as well as z_0, we are led to Cantor sets and fractals and even the Mandelbrot set. (I don't go so far. This is a calculus course, not a graduate course! Do not think that the students need to be geniuses or professional programmers. On the contrary, all they need is the change from $\sqrt{2}$ to $\sqrt{-1}$.) It is not the course that becomes chaotic, just the iteration.

The difficulty in writing about chaos was this. The idea is beautiful to explain, but how is it connected to the rest of calculus? Imagine my pleasure when Newton's method for $x^2 + 1 = 0$ produced exactly what was wanted. It is at the same time an example of iteration and divergence and cycling and chaos.

To see how the iteration works, we find a formula for Newton's x_n. Then comes the connection with z_n, and the particular choice $a = 4$.

A Formula for x_n

Having said that formulas can take over a subject, we now find one. The goal is to match some familiar function with the equation

$$x_{n+1} = \frac{1}{2}\left(x_n - \frac{1}{x_n}\right).$$

The clue is in one of those identities that are exercises in a trigonometry course. They seem to give practice in manipulation and nothing more. But sometimes they fit into another part of mathematics. The identity is

$$\frac{\cos 2\theta}{\sin 2\theta} = \frac{1}{2}\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right) \quad \text{or} \quad \cot 2\theta = \frac{1}{2}\left(\cot\theta - \frac{1}{\cot\theta}\right).$$

In the left equation, the common denominator is $2\sin\theta\cos\theta = \sin 2\theta$. The numerator is $\cos^2\theta - \sin^2\theta = \cos 2\theta$. The identity says this about the iteration:

$$\textit{If } x_0 = \cot\theta \textit{ then } x_1 = \frac{1}{2}\left(x_0 - \frac{1}{x_0}\right) = \cot 2\theta.$$

Then $x_2 = \cot 4\theta$. Then $x_n = \cot 2^n\theta$. **This is the formula.** We are sampling points *along the cotangent curve* (Figure 4). The starting point is at $\theta = \cot^{-1} x_0$.

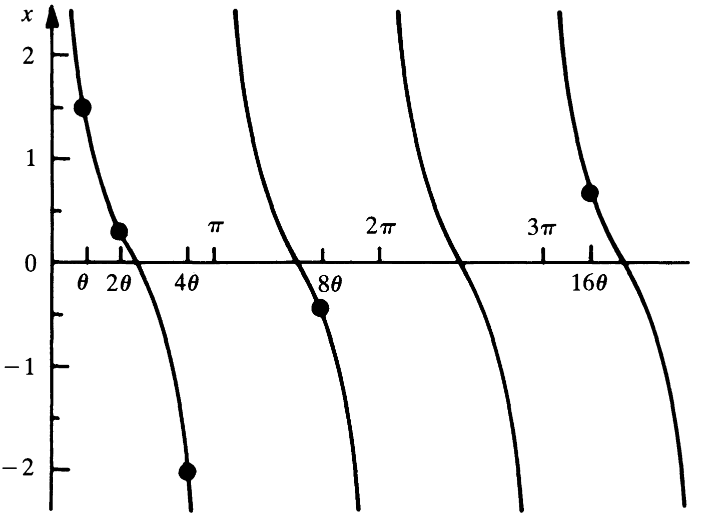


Figure 4
Newton's method searching for i: $x = \cot\theta, \cot 2\theta, \ldots, \cot 2^n\theta$.

Example 1. Start with $\theta = \pi/4$ (cotangent is $x_0 = 1$). The first step gives $\theta = \pi/2$ (cotangent equals 0). The next step is $\theta = \pi$ (iteration blows up).

Example 2. Start with $\theta = \pi/3$ (cotangent is $x_0 = 1/\sqrt{3}$). The first step gives $\theta = 2\pi/3$ (cotangent equals $-1/\sqrt{3}$). The next step is $\theta = 4\pi/3$ (which is the original θ plus π). The iteration cycles (if the arithmetic is exact).

Example 3. Start with a small θ (a large cotangent). After the first step, $\cot 2\theta$ is approximately cut in half (use calculus). The cotangent decreases until the angle $2^n\theta$ passes $\pi/3$. Then the next step makes it larger.

> The iteration eventually blows up if $\theta/\pi = \text{integer}/2^N$.
> The iteration eventually cycles if $\theta/\pi =$ any other fraction p/q.
> The iteration is not periodic (or convergent) if θ/π is irrational.

In the first case, $2^N\theta$ is an integer times π. Its cotangent is infinite. On the graph of $x^2 + 1$, the sequence of tangent lines brought us exactly to zero ($= x_{N-1}$). The tangent at zero is horizontal, so x_N is out at infinity.

In the second case, every angle $2^n\theta$ is an integer times π/q. Within q steps, two of those integers will differ by a multiple of q. So the angles will differ by a multiple of π. Since the cotangent function has period π, the sequence $\{x_n\}$ starts to cycle.

The third case is violently irregular (although determined by a nice formula). There is not just the gentle irregularity of $\cot n\theta$, when two nearby starting points stay reasonably close for a long time. The multiplication $2^n\theta$ moves nearby θ's away from each other exponentially fast. The sequence "forgets" where it started.

To describe the iteration, write θ/π as a number like .100101... (in binary). Then $2\theta/\pi$ has the binary point moved to 1.00101... This integer part 1 makes no difference to the cotangent of 2θ (it changes 2θ by π). Each succeeding step shifts the binary point and drops the integer part.

Note. The zeros and ones tell whether x_n is positive or negative! The first bit is a zero when θ/π is less than $\frac{1}{2}$. Then $\cot\theta$ is positive. The third bit is a one when $2^3\theta/\pi$ (mod 1) is greater than $\frac{1}{2}$. In that case x_3 is negative. The special number .010101... cycles back and forth. *This number is* $1/3$ (in binary)—it produces the example with $\theta = \pi/3$. If we keep only 16 decimals, the cycle is headed for destruction.

Thus the iteration can be analyzed. It is like going around a circle, $\theta \to 2\theta \to 4\theta \to \dots$, coming close to all points but never touching the same point twice. This doubling transformation and its link to Newton's method is already known (it is repeated in *The Beauty of Fractals* [**3**]). But a complete analysis of Newton's method for polynomials of higher degrees is not known. **This is a rich field for experiments**. Of special interest is the "basin of attraction" of each root—the set of starting points x_0 from which Newton's method converges to that root.

In the complex plane those basins are generally **fractals**—a Macintosh is more than enough to draw the figures. (Since roots can be complex numbers, it is natural to allow x_0 to be a complex number.) On the boundaries between basins of attraction, the iterations can't decide which root to converge to. That is where we find chaos. For $x^2+1=0$, starting at $x_0 = a+ib$ leads toward i when b is positive. The limit is $-i$ when b is negative. The basins of attraction are half-planes. The line between basins is the real line, where chaos occurs. Roughly speaking, the iteration cannot make up its mind.

The Connection with Parabolic Iteration

The iteration $x_{n+1} = \frac{1}{2}(x_n - 1/x_n)$ arises naturally from Newton's method. The goal is to bring x^2+1 to zero, which will never happen (for real x). It is interesting to watch the numbers $y_n = x_n^2+1$ at each step—they show how Newton moves on the *vertical* axis. Figure 5 contains the y's, and each new one is

$$y_{n+1} = x_{n+1}^2 + 1 = \frac{1}{4}\left(x_n - \frac{1}{x_n}\right)^2 + 1 = \frac{1}{4}\left(x_n^2 + 2 + \frac{1}{x_n^2}\right).$$

That becomes an iteration for the y_n when we simplify:

$$y_{n+1} = \frac{1}{4}\frac{\left(x_n^2+1\right)^2}{x_n^2} = \frac{1}{4}\frac{y_n^2}{y_n - 1}. \tag{4}$$

This is interesting but not beautiful. The denominator spoils the simplicity. One more step will improve everything—a change to $z = 1/y$. Turn the equation upside down:

$$\frac{1}{y_{n+1}} = \frac{4(y_n-1)}{y_n^2} \quad \text{or} \quad z_{n+1} = 4z_n - 4z_n^2. \tag{5}$$

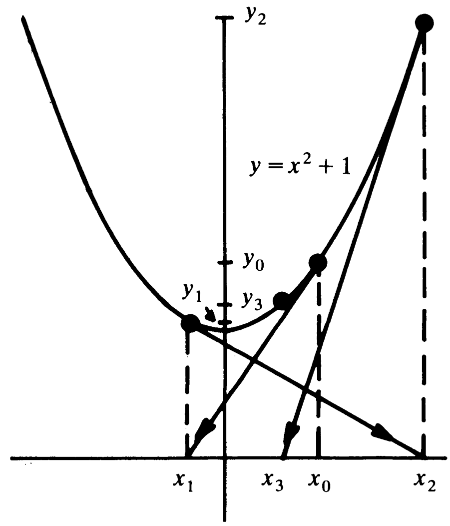


Figure 5
The same points give an iteration for the y's.

Each iteration for the z's involves only the quadratic $4z - 4z^2$. The x's and y's in the denominator are gone. More than that, this is the most studied and most famous of all quadratic iterations **[1]**, **[2]**. It is a discrete analogue of the logistic equation $z' = az - bz^2$, with new properties not shared by the differential equation. The connection between $4z - 4z^2$ and $x^2 + 1 = 0$ is extremely satisfying.

The new variable $z = 1/(x^2 + 1)$ always lies between 0 and 1. The point $x = 0$ corresponds to $z = 1$, and $x = \infty$ corresponds to $z = 0$. The solution $x = \cot 2^n\theta$ gives $y = x^2 + 1 = (\csc 2^n\theta)^2$. Then $z = 1/y = (\sin 2^n\theta)^2$, a neat formula for an important iteration.

Example 1. Starting at $z_0 = \frac{1}{2}$, the function $F(z) = 4z - 4z^2$ gives $z_1 = 1$. Then $z_2 = 0$. All later z's stay at zero (at infinity in the x picture).

Example 2. When the input is $z_0 = \frac{3}{4}$, the output is $z_1 = \frac{3}{4}$. This is a *fixed point*: $\frac{3}{4} = F(\frac{3}{4})$.

Note. Neither of these fixed points 0 and $\frac{3}{4}$ is attractive. The test for convergence to a fixed point uses the most basic idea of calculus: **Compute the slope F' at the point**. The derivative $F' = 4 - 8z$ equals 4 at $z = 0$ and -2 at $z = \frac{3}{4}$. When the absolute value $|F'|$ exceeds 1, as these do, the fixed point is repulsive. The iteration moves away from the point, as we now see.

In practical problems the goal is convergence. Algorithms are constructed to make $|F'|$ as small as possible, since the error is reduced by $|F'|$ at every step. Newton's method is superconvergent because it achieves $F' = 0$. Of course it has to get *near the root*—which never happens for $x^2 = -1$.

Example 3. Start with z_0 small. Then $z_1 = F(z_0)$ is about four times as large, because $F'(0) = 4$. Then z_2 is about $16z_0$. We leave the fixed point at $z = 0$

exponentially fast. But we cannot close in on $z = \frac{3}{4}$, because

$$z_{n+1} - \frac{3}{4} = F(z_n) - F\left(\frac{3}{4}\right) \approx (F')\left(z_n - \frac{3}{4}\right). \tag{6}$$

When F' is -2, a point near $\frac{3}{4}$ is moved twice as far away.

The iteration is seen graphically in Figure 6. Start at (z_0, z_0) on the 45° line. Go up or down to (z_0, z_1) on the curve (its height is $z_1 = F(z_0)$). Then go along to the 45° line at (z_1, z_1), and repeat. The iteration goes from 45° line to parabola to 45° line, producing a "cobweb." The cobweb converges toward a fixed point if $|F'| < 1$. Here the fixed points are repulsive, with slopes 4 and -2, so the iteration has to keep moving on. It can never converge.

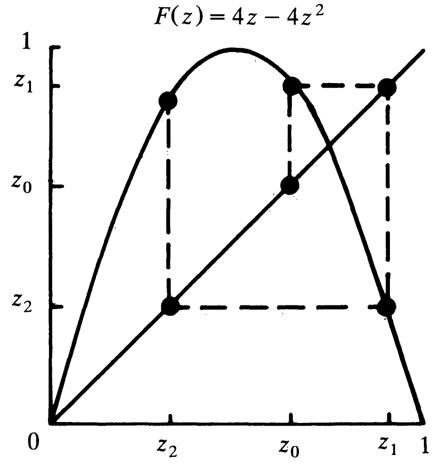


Figure 6
Iteration for z's (cobweb between line and parabola).

The Family of Quadratics

The remaining paragraphs fit the particular function $4z - 4z^2$ into its special position, at the end of the family of quadratics

$$z_{n+1} = az_n - az_n^2, \qquad 0 \leq a \leq 4. \tag{7}$$

It is fascinating to watch the behavior change as a changes. We have seen chaos at the top end $a = 4$. (Above $a = 4$ is divergence to $-\infty$.) Now start with smaller a and see what happens—on a calculator or computer if possible. The boring part is up to $a = 3$, when the z's converge. (Of course applied mathematics tries fervently to stay in this "boring" part. Convergence is exactly what is wanted!) It is easy to find the limit of the z's, as $n \to \infty$ in (7):

$$\text{if } z = az - az^2 \text{ then } z = 0 \quad \text{or} \quad z = \frac{a-1}{a}.$$

Those are the points where $z = F(z)$—the fixed points. The derivatives at those points are $F' = a$ and $F' = 2 - a$. At $a = 4$ we recover $F' = 4$ and $F' = -2$. The convergence test is $|F'| \leq 1$:

for $0 \leq a \leq 1$ the z_n decrease to zero
for $1 \leq a \leq 3$ the z_n approach $(a-1)/a$.

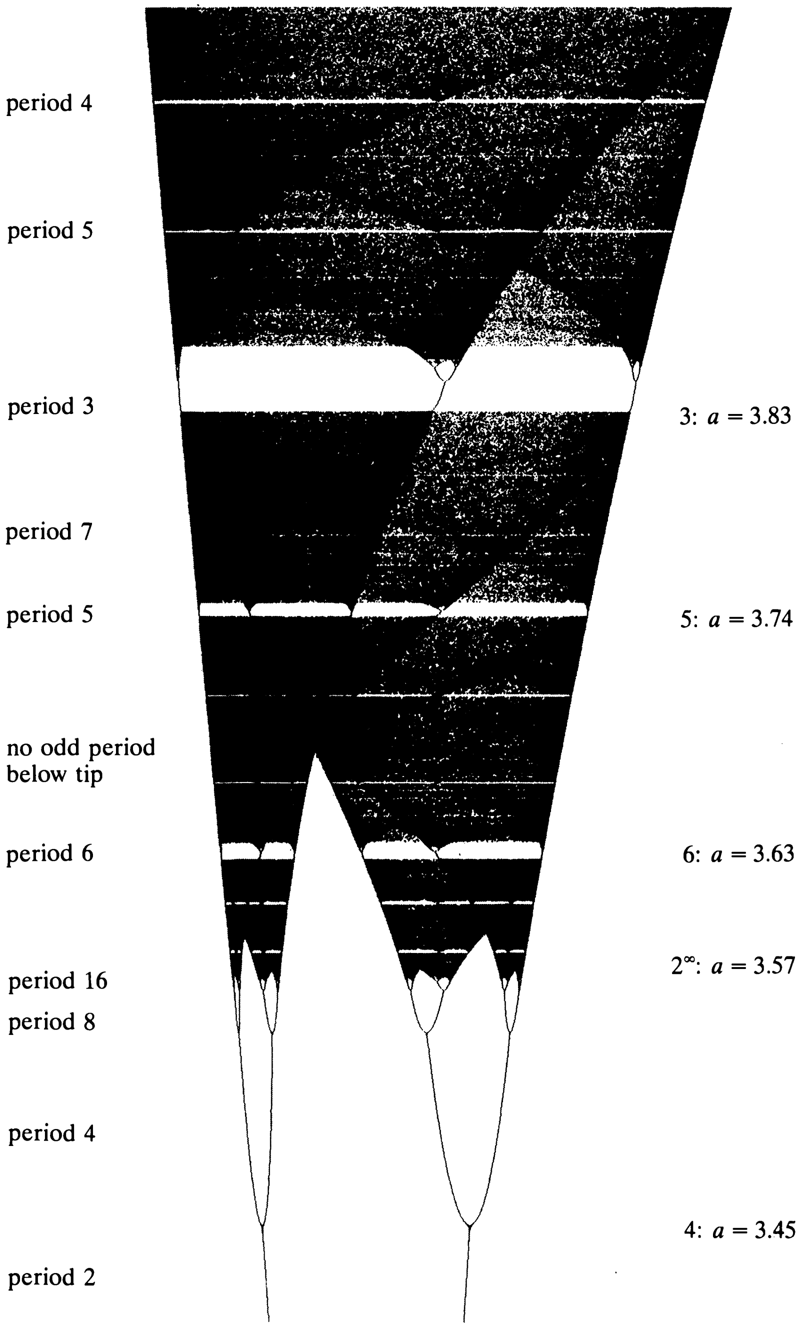


Figure 7

Each row shows a stable cycle or chaos for parameter a (z is across, a is up).

Beyond $a = 3$ convergence is impossible—the test is failed. Cycling is still possible, and the computer shows it happening. I started with a random number z_0, took 100 steps, and printed steps 101 to 105:

	$a = 3.4$	$a = 3.5$	$a = 3.8$	$a = 4.0$
$z_{101} =$	$\underline{.842}$	$\underline{.875}$	.336	.169
$z_{102} =$	.452	.383	.848	.562
$z_{103} =$	$\underline{.842}$	.827	.491	.985
$z_{104} =$	.452	.501	.950	.060
$z_{105} =$	$\underline{.842}$	$\underline{.875}$	.182	.225

The first column is converging to a "2-cycle." It alternates between $x = .842$ and $y = .452$. Those satisfy $y = F(x)$ and $x = F(y) = F(F(x))$. If we look at a *double step* when $a = 3.4$, then both x and y are fixed points of the double iteration $z_{n+2} = F(F(z_n))$. But if a increases past 3.45, this 2-cycle becomes unstable and the z's must go elsewhere.

At that point the period doubles from 2 to 4. When $a = 3.5$ there is a "4-cycle" in the table—it repeats after 4 steps. The sequence bounces from $x = .875$ to $y = F(x) = .383$ to $u = F(y) = .827$ to $v = F(u) = .501$ and back to $x = F(v) = .875$. This cycle must be attractive (also called *stable*) or we would never see it after 100 steps. But it also becomes unstable, in its turn, as a increases. Next comes an 8-cycle, which is stable in a little window around $a = 3.55$. *The stable cycles appear for shorter and shorter intervals of a's*. Those stability windows are shrinking by the recently discovered **Feigenbaum factor** 4.6692... Cycles of length 16, 32, 64 can be seen in physical experiments, but all periods 2^n are unstable before $a = 3.57$. What happens then?

The new and unexpected behavior is between 3.57 and 4. At $a = 3.8$, I have no idea whether the table is showing chaos or part of a long cycle. On each line of Figure 7, the computer has plotted the values of z_{1001} to z_{2000}—omitting the first thousand points to let a stable period (or chaos) become established. No points appeared in the big white wedge. In the window for period 3, around $a = 3.83$, you see only 3 z's. Period 3 is followed by periods $6, 12, 24, \ldots$. There is *period doubling* at the end of every window (including the infinity of windows that are too small to see). This figure can be reproduced, just by iterating the quadratics $z_{n+1} = az_n - az_n^2$ from any starting point and plotting the results.

Eventually we reach $a = 4$ (our own quadratic, at the end of the figure). That is the iteration coming from $x_{n+1} = \frac{1}{2}(x_n - 1/x_n)$, with 2-cycles and 3-cycles and q-cycles (when θ/π is rational). But none of those cycles is attractive, and all other starting points lead to chaos.

Conclusion

This paper went further than I actually go in my calculus class. Certainly Newton's method, and its construction by tangent lines, is essential to the course. (This method is also central to multivariable calculus—it is hard to see why such a key idea is so often missed.) The link between convergence and $|F'| < 1$ is perfect. But the *failure* of Newton's method can be illuminating too. The breakdown stands out with special clarity in the formula $\cot 2^n\theta$. It is remarkable to find so close a relation to the most basic examples of chaotic iteration.

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"There is a complexity in the detail of the prose, like the fractals of modern geometry that replicate natural forms. This sheer accumulation of detail authenticates itself by producing something very like a world".

Marilynne Robinson, *The Guilt She Left Behind*, The New York Times Book Review, April 22, 1990, p. 7 which is a review of Joyce Carol Oates, *Because It Is Bitter, and Because it is My Heart*