Infinite series of complex numbers

Convergence. Given a sequence of complex coefficients $c_0, c_1, c_2, \ldots,$ the series

$$\sum_{k=0}^{\infty} c_k = c_0 + c_1 + c_2 + \ldots + c_n + \ldots$$

is said to converge if the sequence of its partial sums

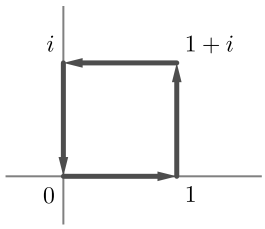
$$s_n = \sum_{k=0}^{n} c_k = c_0 + c_1 + c_2 + \ldots + c_n$$

converges to a finite limit s in the complex plane. The limit $s = \lim_{n\to\infty} s_n$ is then called the sum of the series, and we write

$$\sum_{k=0}^{\infty} c_k = s.$$

Example. The series $\sum_{k=0}^{\infty} i^k$ does not converge (it diverges). The sequence of its partial sums is periodic and therefore does not have a limit:

$$1, \quad 1+i, \quad 1+i+i^2 = i, \quad 1+i+i^2+i^3 = 0, \quad 1+i+i^2+i^3+i^4 = 1, \quad \ldots.$$



Example. Let w be a complex number with $|w| < 1$. Then the series $\sum_{k=0}^{\infty} w^k$ converges and its sum is $\frac{1}{1-w}$. Indeed, the nth partial sum of the series is

$$s_n = \sum_{k=0}^{n} w^k = \frac{1 - w^{n+1}}{1 - w}.$$

Since $|w| < 1$, we have $w^{n+1} \to 0$ $(n \to \infty)$ and so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - w^{n+1}}{1 - w} = \frac{1}{1-w}.$$

The series $\sum_{k=0}^{\infty} w^k$ is called the geometric series with common ratio w.

Necessary condition for convergence. If the series $\sum_{k=0}^{\infty} c_k$ converges, $\lim_{n\to\infty} c_n = 0$.

Proof: $\lim_{n\to\infty} c_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$

Example. For every w with $|w| \geq 1$, the geometric series $\sum_{k=0}^{\infty} w^k$ diverges.

Cauchy's convergence criterion for series. $\sum_{k=0}^{\infty} c_k$ converges if and only if for every $\varepsilon > 0$ there exists an index n_0 (depending on ε) such that

$$|s_{n+m} - s_n| = |c_{n+1} + \ldots + c_{n+m}| < \varepsilon,$$

for every $n > n_0$ and every $m = 1, 2, 3, \ldots$.

Cauchy's convergence criterion for series is a result of applying Cauchy's convergence criterion for sequences to the sequence of partial sums of the series. In practice, this criterion is rarely straightforward to verify, so one usually resorts to various sufficiency tests for convergence.

Tail of the series. The tail or the remainder after nth term is $t_n = \sum_{k=n+1}^{\infty} c_k$.

Clearly, a series converges if and only if any of its tails converges.

The tail sums of a convergent series approach 0.

Indeed, if the series converges and its sum is s, then $t_n = s - s_n \to 0$, as $n \to \infty$.

Example. For $|w| < 1$, the tail of the geometric series $\sum_{k=0}^{\infty} w^k$ is $t_n = \dfrac{w^{n+1}}{1-w}$.

Absolute convergence. $\sum_{k=0}^{\infty} c_k$ is said to converge absolutely if $\sum_{k=0}^{\infty} |c_k| < \infty$.

An absolutely convergent series converges in the usual sense. This is a consequence of the Cauchy convergence criterion, since

$$|c_{n+1} + \ldots + c_{n+m}| \leq |c_{n+1}| + \ldots + |c_{n+m}|, \qquad \text{for every } n \text{ and } m.$$

The converse does not hold.

Triangle inequality for series. If $\sum_{k=0}^{\infty} |c_k| < \infty$, then $\left|\sum_{k=0}^{\infty} c_k\right| \leq \sum_{k=0}^{\infty} |c_k|$.

To prove this, write the triangle inequality for each partial sum, $|s_n| \leq \sum_{k=0}^{n} |c_k|$.

Then, as $n \to \infty$, the left-hand side tends to $|s|$ and the right-hand side tends to $\sum_{k=0}^{\infty} |c_k|$.