Integration of complex functions of real variable

A parameterized curve in $\mathbb{C}$ is a continuous function $\gamma:\ [a,b]\to\mathbb{C}$. Sometimes we will make no distinction between γ and its trace (image), a subset of $\mathbb{C}$.

It is assumed that all integrands below are continuous.

Definition. Given a curve $\gamma(t)=x(t)+iy(t),\ a\le t\le b,$ we set

$$\int_a^b \gamma(t)dt=\int_a^b x(t)dt+i\int_a^b y(t)dt.$$

The quantity $\dfrac{1}{b-a}\displaystyle\int_a^b\gamma(t)dt$ is the average value of $\gamma(t)$ over the interval $[a,\ b]$.

Examples: $\displaystyle\int_0^{2\pi} e^{int}dt=0$ and $\displaystyle\int_0^1 (at+(1-t)ib)dt=\frac{a+ib}{2}$.

Re and Im parts. $\mathrm{Re}\displaystyle\int_a^b\gamma(t)\ dt=\int_a^b \mathrm{Re}\,\gamma(t)\ dt,\ \ \mathrm{Im}\int_a^b\gamma(t)\ dt=\int_a^b\mathrm{Im}\,\gamma(t)\ dt$

Conjugation: $\displaystyle\int_a^b\overline{\gamma(t)}dt=\overline{\int_a^b\gamma(t)dt}$

Linearity: $\displaystyle\int_a^b(c_1\gamma_1(t)+c_2\gamma_2(t))dt=c_1\int_a^b\gamma_1(t)dt+c_2\int_a^b\gamma_2(t)dt,\quad c_1,\ c_2\in\mathbb{C}$

Change of variable. If $t=t(s),\ c\le s\le d,$ is increasing and differentiable, then

$$\int_a^b\gamma(t)dt=\int_c^d\gamma(t(s))t'(s)ds=\int_c^d\beta(s)ds,$$

where $\beta(s)=\gamma(t(s))t'(s),\ c\le s\le d.$ Note that $\gamma(t)$ and $\beta(s)$ generally trace different curves, but the two integrals are equal.

Change of direction. The curve one obtains from $\gamma(t),\ a\le t\le b,$ by reversing its direction can be parameterized by $\gamma_-(t)=\gamma(a+b-t),\ a\le t\le b.$ Note that
$\displaystyle\int_a^b\gamma_-(t)dt=\int_a^b\gamma(a+b-t)dt=\int_b^a\gamma(s)(-ds)=\int_a^b\gamma(s)ds$ (same average position).

Fundamental Theorem of Calculus: $\displaystyle\int_a^b\gamma'(t)dt=\gamma(b)-\gamma(a).$

If $\gamma(t),\ a\le t\le b,$ traces a closed curve, then $\gamma(b)=\gamma(a)$ and so $\displaystyle\int_a^b\gamma'(t)dt=0.$

Arc length. The length of the curve $\gamma(t),\ a\le t\le b,$ is given by $\displaystyle\int_a^b|\gamma'(t)|\,dt.$
When the arc length integral is finite, the path is said to be rectifiable.

Triangle inequality for integrals: $\left|\int_a^b \gamma(t)dt\right| \le \int_a^b |\gamma(t)|dt$

Proof. Let θ be the argument of $\int_a^b \gamma(t)dt$. Then

$$\begin{aligned}\left|\int_a^b \gamma(t)dt\right| &= \mathrm{Re}\left(e^{-i\theta}\int_a^b \gamma(t)dt\right)\\ &= \int_a^b \mathrm{Re}\,(e^{-i\theta}\gamma(t))dt\\ &\le \int_a^b |e^{-i\theta}\gamma(t)|dt = \int_a^b |\gamma(t)|dt.\end{aligned}$$

Equality holds only if $\gamma(t)$ has constant argument. □

Example. $\left|\int_0^\theta (\cos t + i\sin t)dt\right| = |e^{i\theta} - 1|$ and $\int_0^\theta |\cos t + i\sin t|dt = \theta$.

Estimate of the integral. The triangle inequality yields a bound on the integral:

$$\left|\int_a^b \gamma(t)dt\right| \le (b-a)\max_{a\le t\le b}|\gamma(t)|.$$

Uniform convergence. If a sequence of curves $\gamma_n(t)$, $a \le t \le b$, satisfies the condition $\max_{a\le t\le b}|\gamma_n(t) - \gamma(t)| \to 0$, as $n \to \infty$, then $\lim_{n\to\infty}\int_a^b \gamma_n(t)dt = \int_a^b \gamma(t)dt$.

Proof. By the preceding estimate,

$$\begin{aligned}\left|\int_a^b \gamma_n(t)dt - \int_a^b \gamma(t)dt\right| &= \left|\int_a^b (\gamma_n(t)dt - \gamma(t))dt\right|\\ &\le (b-a)\max_{a\le t\le b}|\gamma_n(t) - \gamma(t)| \to 0, \quad n \to \infty. \qquad \square\end{aligned}$$

References.

Stephen Fisher, *Complex variables.*

Donald Sarason, *Complex Function Theory.*