The limit of circular averages

Let f be a continuous function on an open set G and let z_0 be a point of G. For small enough $r > 0$, let C_r be the circle $|z - z_0| = r$, with the counterclockwise orientation, contained in G together with its interior. Then

$$\lim_{r\to 0} \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-z_0}\,dz \;=\; \lim_{r\to 0} \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta \;=\; f(z_0).$$

Proof. Parameterize $C_r:\ z(t) = z_0 + re^{it},\ 0 \le t \le 2\pi$. Then

$$\frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{re^{it}}\,rie^{it}dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})dt.$$

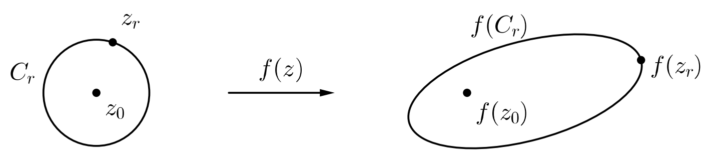
Therefore

$$\left|\frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-z_0}\,dz - f(z_0)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi} (f(z_0+re^{it}) - f(z_0))dt\right| \le \max_{z\in C_r} |f(z) - f(z_0)|\ .$$

The conclusion now follows since

$$\max_{0\le t\le 2\pi} |f(z_0+re^{it}) - f(z_0)| \to 0, \quad \text{as } r \to 0.$$

To justify the last claim, let $z_r = z_0 + re^{it_r}$ be a point on C_r such that $f(z_r)$ is a most distant from $f(z_0)$ point of the curve $f(C_r)$. Note that, for each r, the maximum distance is attained (perhaps, more than once), because $f(z)$ is continuous on C_r.



Then $z_r \to z_0$ as $r \to 0$. So, by continuity of f at z_0, $f(z_r) \to f(z_0)$. □

Remark. When f is holomorphic in G, the circular averages are independent of r:

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta \;=\; f(z_0).$$

This is a consequence of Cauchy's theorem.