Differential Calculus
Grinshpan

The art of finding limits, part I

RULES

\[
\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

\[
\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)
\]

\[
\lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \text{if } \lim_{x \to a} g(x) \neq 0.
\]

The rules apply if both \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist. Here is how they work:

\[
\lim_{x \to 3} \frac{2x - 1}{x^2 + x} = \frac{\lim_{x \to 3} (2x - 1)}{\lim_{x \to 3} (x^2 + x)} = \frac{2 \lim_{x \to 3} x - \lim_{x \to 3} 1}{\lim_{x \to 3} x^2 + \lim_{x \to 3} x} = \frac{2 \times 3 - 1}{3^2 + 3} = \frac{5}{12}
\]

CONTINUITY

If \( f(x) \) is known to be continuous at \( a \), then \( \lim_{x \to a} f(x) = f(a) \).

\[
\lim_{x \to 47.8} e^x = e^{47.8}
\]

ALGEBRA

We use algebra to resolve inconclusive behavior, until the limit rules become applicable. Inconclusive behaviors have code names such as \( \frac{0}{0}, \infty - \infty, \frac{\infty}{\infty}, 0 \times \infty \).

\[
\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x - 2} = \lim_{x \to 1} \frac{(x + 1)(x - 1)}{(x + 1)(x - 2)} = \lim_{x \to 1} \frac{x - 1}{x - 2} = \frac{2}{3}
\]

\[
\lim_{x \to \infty} \frac{2x + 1}{x - 3} = \lim_{x \to \infty} \frac{2 + 1/x}{1 - 3/x} = \frac{2 + 0}{1 - 0} = 2
\]

ANALYSIS

Limits submit to patient reason. This is particularly true for compositions.

\[
\lim_{x \to \frac{\pi}{2}^+} e^{\sec x} = ? \quad \text{As } x \text{ tends to } \pi/2 \text{ from the right,}
\]

\[
\cos x \text{ tends to } 0 \text{ from the left,}
\]

\[
\text{so } \sec x = \frac{1}{\cos x} \text{ tends to } -\infty,
\]

\[
\text{so } e^{\sec x} \text{ tends to zero.}
\]

NO LIMIT

The limit may not exist.

\[
\lim_{x \to 0} \frac{|x|}{x} \text{ does not exist: one-sided limits are unequal}
\]

\[
\lim_{x \to 6^+} \frac{1}{6 - x} \text{ does not exist: as } x \text{ tends to 6 from the right,}
\]

\[
6 - x \text{ tends to zero from the left,}
\]

\[
\text{so } \frac{1}{6 - x} \text{ tends to } -\infty.
\]