Lines and circles in $\mathbb{C}$

Three distinct points $z,\, z_1,\, z_2 \in \mathbb{C}$ are on one line whenever the argument of

$$\frac{z-z_1}{z-z_2}$$

is either 0 or π, i.e., the above ratio is real:

$$\frac{z-z_1}{z-z_2} = \frac{\overline{z}-\overline{z}_1}{\overline{z}-\overline{z}_2}.$$

This is equivalent to the equality

$$(\overline{z}_1-\overline{z}_2)z-(z_1-z_2)\overline{z}+z_1\overline{z}_2-\overline{z}_1z_2=0.$$

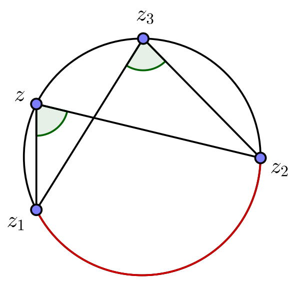
Thus the lines in the complex plane are described by the equations of the form

$$az-\overline{az}+b=0,$$

where $a\in\mathbb{C}$ and b is necessarily purely imaginary.

Four distinct points $z,\, z_1,\, z_2,\, z_3 \in \mathbb{C}$ are on one circle (perhaps, of infinite radius) whenever

$$\arg\,\frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2} = 0 \text{ or } \pi.$$



Reorganizing the terms of the equality

$$\frac{z-z_1}{z-z_2} : \frac{z_3-z_1}{z_3-z_2} = \frac{\overline{z}-\overline{z}_1}{\overline{z}-\overline{z}_2} : \frac{\overline{z}_3-\overline{z}_1}{\overline{z}_3-\overline{z}_2},$$

we find that z satisfies an equation of the form

$$c|z|^2+az-\overline{az}+b=0,$$

where $a\in\mathbb{C}$ and $b,\, c$ are necessarily purely imaginary.[1]
This is the general equation of a circle in the complex plane. It reduces to the equation of a line when $c=0$. This equation can also be written as

$$\alpha|z|^2+\beta\,\mathrm{Re}\,z+\gamma\,\mathrm{Im}\,z+\delta=0 \quad \text{or} \quad \alpha(x^2+y^2)+\beta x+\gamma y+\delta=0,$$

where $\alpha,\, \beta,\, \gamma,\, \delta$ are real coefficients.

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[1] Here $c=2i\,\mathrm{Im}\,(\overline{z}_1-\overline{z}_3)(z_2-z_3),\; a=\overline{c}z_1-c\overline{z}_2,\; b=2i\,\mathrm{Im}\,(cz_1\overline{z}_2).$