**Local Limit Theorem**

Let $X$ be a random variable taking integer values, with mean $\mu$ and variance $\sigma^2$.

Consider a sequence of independent identical copies of $X$, $X_1, X_2, X_3, \ldots$ (repetitions of the same experiment). Then $X_i$ have the same probability mass function, and

$$E[X_i] = \mu, \quad \text{Var}(X_i) = \sigma^2, \quad i = 1, 2, 3, \ldots.$$ 

For each $n$, the sum $X_1 + \ldots + X_n$ is an integer-valued random variable with mean

$$E[X_1 + \ldots + X_n] = E[X_1] + \ldots + E[X_n] = \mu + \ldots + \mu = n\mu$$

and variance

$$\text{Var}(X_1 + \ldots + X_n) \overset{\text{ind}}{=} \text{Var}(X_1) + \ldots + \text{Var}(X_n) = \sigma^2 + \ldots + \sigma^2 = n\sigma^2.$$ 

We would like to estimate the probability that $X_1 + \ldots + X_n = k$ for large values of $n$. (For small values of $n$ this probability can often be found exactly.) In these matters local limit theorems can be helpful.

**Theorem** For large $n$, $P(X_1 + \ldots + X_n = k) \approx \frac{1}{\sqrt{2\pi n}} e^{-(k-n\mu)^2/2n\sigma^2} \frac{1}{\sigma \sqrt{n}}$.

In terms of the values $x_{k,n} = \frac{k-n\mu}{\sigma \sqrt{n}}$ of standardized sums $S_n^* = \frac{X_1 + \ldots + X_n - n\mu}{\sigma \sqrt{n}}$, the theorem asserts that

$$P(S_n^* = x_{k,n}) \approx \frac{\varphi(x_{k,n})}{\sigma \sqrt{n}} \quad (n \text{ large}),$$

where $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the standard normal density.

**Example** [Grinstead & Snell] A bank accepts rolls of pennies and gives 50 cents credit to a customer without counting the contents. Assume that a roll contains 49 pennies 30 percent of the time, 50 pennies 60 percent of the time, and 51 pennies 10 percent of the time. Estimate the probability that the bank will lose exactly 25 cents in 100 rolls.

Let us introduce a random variable $X$ representing the difference between the actual number of pennies in a roll and 50. Then, based on the assumptions, $X$ has the following probability mass function

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_X$</td>
<td>.3</td>
<td>.6</td>
<td>.1</td>
</tr>
</tbody>
</table>

The expected value of $X$ is $\mu = (-1) \times .3 + 0 \times .6 + 1 \times .1 = -\frac{1}{3}$ and its variance is $\sigma^2 = E[X^2] - E^2[X] = .4 - .04 = .36$.

Let $X_1, X_2, \ldots, X_{100}$ be independent identical copies of $X$ corresponding to 100 rolls. The question is to estimate the probability that $X_1 + \ldots + X_{100} = -25$. We have $n = 100$ and

$$\frac{-25 - n\mu}{\sigma \sqrt{n}} = -\frac{5}{6}.$$ 

So the local limit theorem suggests that the probability that $X_1 + \ldots + X_{100} = -25$ is close to $\varphi(-\frac{5}{6})/6 \approx .047$. 