MIXING PROBLEMS. EXAMPLES.

Consider the following setup. A solution of salt and water is poured into a tank containing some salty water and then poured out. It is assumed that the incoming solution is instantly dissolved into a homogeneous mix. Given are the constant parameters:

\( V_0 \), the original capacity of salty solution in the tank (in gallons)
\( A_0 \), the original amount of salt in the tank (in pounds)
\( R_{\text{in}} \), the incoming rate of salty solution (in gallons per minute)
\( C \), the concentration of salt in the incoming solution (in pounds per gallon)
\( R_{\text{out}} \), the outcoming rate of salty solution (in gallons per minute)

The amount \( A(t) \) of salt in the tank at time \( t \) needs to be determined.

EQUATION. The actual salt is coming in at a rate of \( CR_{\text{in}} \) (pounds per gallon) and coming out at a rate of \( \frac{A(t)}{V(t)} R_{\text{out}} \) (pounds per gallon), where \( V(t) = V_0 + t(R_{\text{in}} - R_{\text{out}}) \) (gallons) is the volume of fluid in the tank at time \( t \). Hence the differential equation describing the variation of salt mass is

\[
\frac{d}{dt} A(t) = CR_{\text{in}} - \frac{A(t)}{V_0 + t(R_{\text{in}} - R_{\text{out}})} R_{\text{out}}, \quad A(0) = A_0.
\]

If \( R_{\text{in}} = R_{\text{out}} = R \), the equation has a particularly simple form as it has constant coefficients. In this case the solution is given by

\[
A(t) = CV_0 + (A_0 - CV_0)e^{-\frac{R}{V_0} t}.
\]

Note that \( A(t) \to CV_0 \) as \( t \to \infty \).

If \( R_{\text{in}} \neq R_{\text{out}} \), the equation can always be solved by using an integrating factor. In this case the solution is only valid until the tank will overflow or run out of fluid.

EXAMPLE 1. A tank initially contains 200 gal of pure water. Beginning at time \( t = 0 \), brine containing 5 pounds of salt per gallon is added to the tank at a rate of 20 gal/min and then the mixed solution is drained from the tank at the same rate. How much salt is in the tank after 30 minutes?

SOLUTION. Let \( A(t) \) be the amount of salt in the tank at time \( t \), \( A(0) = 0 \). According to our template,

\[
A(t) = 1000(1 - e^{-\frac{1}{20} t}), \quad t \geq 0.
\]

In particular, \( A(30) = 1000(1 - e^{-3}) \approx 950 \) (lb).

EXAMPLE 2. A 500-gallon tank originally contains 100 gallons of fresh water. Beginning at time \( t = 0 \), water containing 50% pollutants flows into the tank at a rate of 2 gal/min and then the well-stirred solution is drained at a rate of 1 gal/min. Find the concentration of pollutants in the tank at the moment it overflows.
SOLUTION. Let $A(t)$ be the mass of pollutants at time $t$, $A(0) = 0$. If $t_0$ is the moment of overflow, then the desired concentration is $A(t_0)/500$. We have the differential equation

$$\frac{d}{dt} A(t) = 1 - \frac{A(t)}{100 + t}, \quad A(0) = 0.$$ 

Hence

$$(100 + t) \frac{dA}{dt} + A = 100 + t$$

$$(100 + t)A = 100t + \frac{1}{2} t^2 + c, \quad c = 0$$

$$A(t) = \frac{100 + \frac{1}{2} t}{100 + t} t$$

This solution is only valid for $0 \leq t \leq t_0$. Since $500 = 100 + t_0(2 - 1)$, we conclude that $t_0 = 400$. Thus $A(400)/500 = \frac{12}{25}$ or 48% is the answer.