THE P-SERIES

The series
$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$
is called the p-series. Its sum is finite for $p > 1$ and is infinite for $p \leq 1$.
If $p = 1$ we have the harmonic series.

For $p > 1$, the sum of the p-series (the Riemann zeta function $\zeta(p)$) is a monotone decreasing function of p.

For almost all values of p the value of the sum is not known. For instance, the exact value of the sum $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a mystery. But, of course, one can always find accurate approximations for any given p.

Some of the known sums and approximations are
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2020569$$
$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \sum_{n=1}^{\infty} \frac{1}{n^5} \approx 1.0369278$$
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \qquad \sum_{n=1}^{\infty} \frac{1}{n^7} \approx 1.0083493$$
One often compares to a p-series when using the Comparison Test.

Example. Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ for convergence.
Solution. Observe that
$$\frac{1}{n^2+3} < \frac{1}{n^2}$$
for every $n \geq 1$. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$). So the given series converges too, by the Comparison Test.

Or when using the Limit Comparison Test.

Example. Test the series $\sum_{n=1}^{\infty} \frac{n}{n^{3/2}+3}$ for convergence.
Solution. Observe that
$$\frac{n}{n^{3/2}+3} : \frac{1}{\sqrt{n}} = \frac{n^{3/2}}{n^{3/2}+3} = \frac{1}{1+3n^{-3/2}} \to 1 \neq 0, \quad n \to \infty.$$
The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series with $p = \frac{1}{2} \leq 1$). So the given series diverges as well, by the Limit Comparison Test.