Physical interpretation of differentiability

Not every complex function $f(z) = u + iv$ has the complex derivative,

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}.$$

Analytically, differentiability is expressed by the Cauchy-Riemann equations,

$$v_x = -u_y, \quad v_y = u_x.$$

Geometrically, it means that $f(z)$ preserves angles at any point z with $f'(z) \neq 0$.

There is a physical interpretation as well.

Consider the vector field $\vec{F} = (u, -v)$, the velocity or intensity of a planar flow.

The quantity $u_x + (-v)_y$ is the divergence of $\vec{F}$, it measures the outflow density at a point. If the divergence is positive, the point acts a source, and if the divergence is negative, the point acts as a sink. The Cauchy-Riemann equation

$$u_x + (-v)_y = 0$$

states that the divergence of $\vec{F}$ is zero at every point, i.e., the vector field is sourceless. For instance, the field of steady flow of an incompressible fluid is sourceless.

The quantity $(-v)_x - u_y$ is the curl of $\vec{F}$, it measures the extent of rotation of the flow at a point. Equivalently, it measures the work per unit area done by the vector field when a test particle makes an infinitesimal loop around the point. In a conservative field of force, the work along any closed curve is zero. The Cauchy-Riemann equation

$$(-v)_x - u_y = 0$$

states that the curl of $\vec{F}$ is zero at every point, i.e., the vector field is irrotational.

Thus a differentiable function of a complex variable $f(z) = u + iv$ is represented by a sourceless and irrotational field $\vec{F} = (u, -v)$.

Example. The function

$$f(z) = \frac{1}{z} = \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2}$$

is differentiable for $z \neq 0$, and $f'(z) = -1/z^2$. The associated vector field $\vec{F} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ is sourceless and irrotational away from the origin:

$$\operatorname{div} \vec{F} = u_x + (-v)_y = \frac{x^2 + y^2 - 2x^2}{(x^2+y^2)^2} + \frac{x^2 + y^2 - 2y^2}{(x^2+y^2)^2} = 0,$$

$$\operatorname{curl} \vec{F} = (-v)_x - u_y = \frac{-2xy}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2} = 0.$$

At the origin, $\vec{F}$ flows outward radially: this is the only source of the field.

References

Stephen Fisher, *Complex Variables*

George Polya & Gordon Latta, *Complex Variables*