Calculus Notes

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SERIES WITH POSITIVE TERMS

Let

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots + a_n + \ldots$$

be a series with positive terms. The sum of positive terms is always greater than 0. The partial sums

$$S_n = \sum_{k=1}^{n} a_k = a_1 + \ldots + a_n, \quad n = 1, 2, 3, \ldots,$$

are positive and form a monotone increasing sequence:

$$S_{n+1} = S_n + a_{n+1} > S_n, \quad n = 1, 2, 3, \ldots.$$

There can be two scenarios.

Scenario 1: The partial sums S_n are bounded above. Then $\lim_{n\to\infty} S_n = S$ exists by the Monotone Sequence Theorem, and so the series converges to a finite positive value,

$$\sum_{n=1}^{\infty} a_n = S < \infty.$$

Scenario 2: The partial sums S_n increase without bound. Then $\lim_{n\to\infty} S_n = \infty$ and so the series diverges to infinity,

$$\sum_{n=1}^{\infty} a_n = \infty.$$

Thus for a series with positive terms, the issue of convergence/divergence is an issue of finiteness/infiniteness. This is not the case for series in general. For instance, the series

$$1 - 1 + 1 - 1 + 1 - \ldots$$

diverges not because it is infinite in value, but because the sequence of its partial sums $1, 0, 1, 0, \ldots$ does not have a limit, neither finite nor infinite.

Remark. Sometimes one wants to bound the sum rather than find its exact value (which could be tricky or impossible). If $S_n \le M$ for every n, i.e. the first scenario takes place, then $\lim_{n\to\infty} S_n \le M$ and so

$$a_1 + a_2 + \ldots + a_n + \ldots \le M.$$