Differentiation of power series

The series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} nc_n(z-z_0)^{n-1}$ have the same radius of convergence.

Theorem. Let the power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ represent $f(z)$ in $|z-z_0|<R$.
Then $f(z)$ is holomorphic in $|z-z_0|<R$ and $f'(z)=\sum_{n=1}^{\infty} nc_n(z-z_0)^{n-1}$.

Proof. We may assume that $z_0=0$ (otherwise consider $\tilde{z}=z-z_0$).

Fix w with $|w|<R$. We will show that $f'(w)=\lim_{z\to w}\frac{f(z)-f(w)}{z-w}=\sum_{n=1}^{\infty} nc_n w^{n-1}$.

Note that for each $n\geq 2$,

$$\begin{aligned}\frac{z^n-w^n}{z-w} - nw^{n-1} = \sum_{k=0}^{n-1} z^k w^{n-1-k} - nw^{n-1} &= \sum_{k=1}^{n-1}(z^k w^{n-1-k}-w^{n-1})\\ &= \sum_{k=1}^{n-1} w^{n-1-k}(z^k-w^k)\\ &= \sum_{k=1}^{n-1} w^{n-1-k}\sum_{j=0}^{k-1} z^j w^{k-1-j}(z-w).\end{aligned}$$

Choose r with $|w|<r<R$. Then, for each z with $|z|<r$,

$$\left|\frac{z^n-w^n}{z-w} - nw^{n-1}\right| < \sum_{k=1}^{n-1} r^{n-1-k}\sum_{j=0}^{k-1} r^j r^{k-1-j}|z-w| = \tfrac{1}{2}\,n(n-1)\,r^{n-2}|z-w|.$$

Consequently the modulus of the difference

$$\frac{f(z)-f(w)}{z-w} - \sum_{n=1}^{\infty} nc_n w^{n-1} = \sum_{n=2}^{\infty} c_n\left[\frac{z^n-w^n}{z-w} - nw^{n-1}\right]$$

is bounded by

$$\frac{1}{2}\sum_{n=2}^{\infty} n(n-1)|c_n|\,r^{n-2}\;|z-w| = M|z-w|.$$

The power series $\sum_{n=2}^{\infty} n(n-1)c_n z^{n-2}$ converges absolutely for $|z|<R$, so $M<\infty$.

Therefore $\lim_{z\to w}\left|\frac{f(z)-f(w)}{z-w} - \sum_{n=1}^{\infty} nc_n w^{n-1}\right| \leq M\lim_{z\to w}|z-w| = 0.$ □